The geometry of $M_{1,3}$

Tiago J. Fonseca

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1 Introduction

This is joint work with Francis Brown. The goal of the talk is to prove that the motive of the moduli space $M_{1,3}$ is mixed Tate.

Let us fix once and for all a base field k of characteristic zero over which every scheme or stack will be defined. All stacks here are Deligne-Mumford stacks. We denote by $\mathcal{M}_{1,n}$ the moduli stack classifying smooth genus 1 curves with n marked points $(C, p_1, \ldots, p_n)$, and by $M_{1,n}$ its coarse moduli space.

2 Elliptic curves

Recall that an *elliptic curve* is a smooth genus 1 curve with 1 marked point. By abuse, we denote it simply by E. Note that $\mathcal{M}_{1,1}$ is the moduli stack that classifies elliptic curves.

Example 1 (Weierstrass equations)**.** Given $g_2, g_3 \in k$ with $\Delta := g_2^3 - 27g_3^2 \neq 0$, we define an elliptic curve E_{g_2,g_3} to be the projective plane curve defined by the affine equation

$$y^2 = 4x^3 - g_2 x - g_3$$

with marked point the point at infinity $(0 : 1 : 0)$.

The following theorem summarizes classical results on elliptic curves.

Theorem 1. *Every elliptic curve is isomorphic to some E_{g_2,g_3}. Moreover:*

(i) Every isomorphism $E_{g_2,g_3} \xrightarrow{\sim} E_{g_2',g_3'}$ is of the form

$$(x, y) \mapsto (u^2 x, u^3 y),$$

with $g_2' = u^4 g_2$ and $g_3' = u^6 g_3$.

(ii) E_{g_2,g_3} and $E_{g_2',g_3'}$ are isomorphic if and only if

$$\frac{g_2^3}{g_2^3 - 27g_3^2} = \frac{{g_2'}^3}{{g_2'}^3 - 27{g_3'}^2}.$$

The number $j = g_2^3/\Delta$ is called the j-invariant *of E_{g_2,g_3}.*

(iii) Every element of k is the j-invariant of some elliptic curve.

An immediate corollary of the above result is that

$$\mathcal{M}_{1,1} \cong \left[\frac{\mathbb{A}^2 \setminus V(\Delta)}{\mathbb{G}_m}\right],$$

where the quotient in brackets means 'quotient stack' and $\mathbb{A}^2 = \operatorname{Spec} k[g_2, g_3]$ has the $\mathbb{G}_m$-action for which g_2 has weight 4 and g_3 has weight 6. The coarse moduli space is given by

$$j : \mathcal{M}_{1,1} \to \mathbb{A}^1 \cong M_{1,1}.$$

Remark 1. The moduli stacks $\mathcal{M}_{1,n}$ are representable by schemes only for $n \geq 5$: the involution $(x, y) \mapsto (x, -y)$ always fixes the point at infinity and the three points where $y = 0$.

3 Geometry of $M_{1,n}$

The moduli space $M_{1,n}$ is an algebraic variety of dimension n. One might ask what its geometry looks like.

Theorem 2 (Belorousski '98)**.** *$M_{1,n}$ is rational for $n \leq 10$.*

This bound on n is sharp, as one can show that the existence of a cuspidal modular form of weight 12 implies that the Deligne-Mumford compactification $\overline{M}_{1,11}$ admits a global 11-form.

A cohomology computation, or a point count over finite fields, suggests the following.

Conjecture 1. *The motive of $M_{1,n}$ is mixed Tate for $n \leq 10$.*

Loosely speaking, this means that the cohomology of $M_{1,n}$, in *any* Weil cohomology theory, is as simple as possible, in the sense that it can be constructed out of the cohomology of projective spaces.

Formally, the motive of $M_{1,n}$, which is the same as the motive of $\mathcal{M}_{1,n}$, is regarded as an object in Voevodsky's triangulated category of motives. I'll not explain what a motive or a mixed Tate motive is. It suffices to assume the following:

(i) The motive of a quotient stack of the form $[\mathbb{A}^n/G]$, where G is a finite group, is mixed Tate.

(ii) Let $\mathcal{X}$ be a stack and $\mathcal{Z}$ be a closed substack. If the motive of any two of the three stacks $\mathcal{X}, \mathcal{Z}, \mathcal{X} \setminus \mathcal{Z}$ is mixed Tate, then the motive of the third is also mixed Tate.

Example 2. Equip $\mathbb{A}^{n+1} = \operatorname{Spec} k[x_0, \dots, x_n]$ with the action by the multiplicative group $\mathbb{G}_m$ for which x_i has weight d_i. Then the motive of the *weighted projective stack*

$$\mathcal{P}(d_0, \dots, d_n) = \left[\frac{\mathbb{A}^{n+1} \setminus \{0\}}{\mathbb{G}_m}\right]$$

is mixed Tate, since the open substack where $x_i \neq 0$ is isomorphic to a quotient $[\mathbb{A}^n/\mu_{d_i}]$ and its complement is a weighted projective stack of smaller dimension.

In the rest of the talk, I'll exhibit explicit stratifications which, by the properties above, will prove the conjecture for $n \leq 3$.

Remark 2. The cases $n = 1$ and $n = 2$ are well-known. The case $n = 3$ seems to be new.

4 Stratification of $\mathcal{M}^*_{1,3}$

Instead of working directly with $\mathcal{M}_{1,n}$, we shall consider a 'partial compactification' $\mathcal{M}^*_{1,n}$ which will have a nicer description. The stack $\mathcal{M}^*_{1,n}$ classifies genus 1 curves with n marked points $(C, p_1, \dots, p_n)$, as before, but now C is either smooth or a nodal cubic. It is enough to prove that the motive of $\mathcal{M}^*_{1,n}$ is mixed Tate, because $\mathcal{M}_{1,n}$ is the complement of a configuration space of points on a nodal cubic, the motive of which is known to be mixed Tate.

A nodal cubic is given by a Weierstrass equation $y^2 = 4x^3 - g_2x - g_3$ with $\Delta = 0$ but $(g_2, g_3) \neq (0, 0)$. Thus,

$$\mathcal{M}^*_{1,1} = \left[\frac{\mathbb{A}^2 \setminus \{0\}}{\mathbb{G}_m}\right] = \mathcal{P}(4, 6)$$

is a weighted projective stack. One can see it has the mixed Tate property either by stratifying it, or by noting that its coarse moduli space is $\mathbb{P}(4, 6) \cong \mathbb{P}^1$.

Lemma 1. *There are isomorphisms*

$$\mathcal{M}^*_{1,2} \cong \left[\frac{\mathbb{A}^3 \setminus V(g_2, y^2 - 4x^3)}{\mathbb{G}_m}\right] \cong \mathcal{P}(2, 3, 4) \setminus \operatorname{Spec} k,$$

where $\mathbb{A}^3 = \operatorname{Spec} k[x, y, g_2]$*, with* x, y, g_2 *of weights* $2, 3, 4$*.*

Proof. A point (C, p_1, p_2) of $\mathcal{M}^*_{1,2}$ corresponds to a $\mathbb{G}_m$-orbit of a tuple (x, y, g_2, g_3) satisfying $y^2 = 4x^3 - g_2x - g_3$ and $(g_2, g_3) \neq (0, 0)$. Since the Weierstrass equation can be solved for g_3, we can write

$$\mathcal{M}^*_{1,2} = \left[\frac{\mathbb{A}^3 \setminus V(g_2, y^2 - 4x^3)}{\mathbb{G}_m}\right] = \left[\frac{\mathbb{A}^3 \setminus \{0\}}{\mathbb{G}_m}\right] \setminus \left[\frac{V(g_2, y^2 - 4x^3) \setminus \{0\}}{\mathbb{G}_m}\right]$$

By normalizing the cuspidal cubic $y^2 = 4x^3$ by $\mathbb{A}^1 = \operatorname{Spec} k[z]$ via $z \mapsto (z^2, 2z^3)$, we obtain

$$\left[\frac{V(g_2, y^2 - 4x^3) \setminus \{0\}}{\mathbb{G}_m}\right] \cong \left[\frac{\operatorname{Spec} k[z, z^{-1}]}{\mathbb{G}_m}\right] = \operatorname{Spec} k. \qquad \square$$

Theorem 3. *There is a closed substack $\mathcal{Z}$ of $\mathcal{M}^*_{1,3}$ such that $\mathcal{Z} \cong \mathcal{M}^*_{1,2} \setminus \mathcal{P}(2,4)$ and $\mathcal{M}^*_{1,3} \setminus \mathcal{Z} \cong [\mathbb{A}^3/\mu_2] \setminus W$, where W is a scheme with a stratification of the form $W \cong (\mathbb{A}^1 \setminus \{-1,0,1\}) \cup \operatorname{Spec} k \cup \operatorname{Spec} k$.*

Proof. A point (C, p_1, p_2, p_3) in $\mathcal{M}^*_{1,3}$ corresponds to a $\mathbb{G}_m$-orbit of a tuple $(x_1, y_1, x_2, y_2, g_2, g_3)$, where $y_i^2 = 4x_i^3 - g_2 x_i - g_3$, $(x_1, y_1) \neq (x_2, y_2)$, and $(g_2, g_3) \neq (0,0)$. Let $\mathcal{Z}$ be the closed substack where $x_1 = x_2$, and $\mathcal{U}$ be its complement.

On $\mathcal{Z}$, the Weierstrass equations imply that $y_1^2 = y_2^2$, and hence the condition $(x_1, y_1) \neq (x_2, y_2)$, amounts to $y_1, y_2 \neq 0$ and $y_2 = -y_1$. In particular, the data of the pair of points (x_1, y_1), (x_2, y_2) is uniquely determined by (x_1, y_1), and hence

$$\mathcal{Z} \cong \mathcal{M}^*_{1,2} \setminus V(y) \cong \mathcal{M}^*_{1,2} \setminus \mathcal{P}(2,4),$$

where the last isomorphism follows from the previous lemma.

The Weierstrass equations for (x_i, y_i) can be written in matrix form:

$$\begin{pmatrix} x_1 & 1 \\ x_2 & 1 \end{pmatrix} \begin{pmatrix} g_2 \\ g_3 \end{pmatrix} = \begin{pmatrix} -y_1^2 + 4x_1^3 \\ -y_2^2 + 4x_2^3 \end{pmatrix}.$$

On $\mathcal{U}$, they may be uniquely solved for (g_2, g_3), and hence

$$\mathcal{U} \cong \left[\frac{\operatorname{Spec} k[x_1, y_1, x_2, y_2, (x_1 - x_2)^{-1}] \setminus V(y_1^2 - 4x_1^3, y_2^2 - 4x_2^3)}{\mathbb{G}_m} \right].$$

Since x_1, x_2 have weight 2 for the action of $\mathbb{G}_m$, slicing the quotient by $x_1 - x_2 = 1$ yields

$$\left[\frac{\operatorname{Spec} k[x_1, y_1, x_2, y_2, (x_1 - x_2)^{-1}]}{\mathbb{G}_m} \right] \cong \left[\frac{\operatorname{Spec} k[x_1, y_1, y_2]}{\mu_2} \right] \cong \left[\frac{\mathbb{A}^3}{\mu_2} \right].$$

By normalizing the cuspidal cubics $y_i^2 = 4x_i^3$ and deleting $V(x_1 x_2)$, we have furthermore

$$\begin{aligned} \left[\frac{V(y_1^2 - 4x_1^3, y_2^2 - 4x_2^3) \setminus V(x_1 x_2 (x_1 - x_2))}{\mathbb{G}_m} \right] &\cong \left[\frac{\operatorname{Spec} k[z_1^{\pm}, z_2^{\pm}] \setminus V(z_1^2 - z_2^2)}{\mathbb{G}_m} \right] \\ &\cong \mathbb{A}^1 \setminus \{1, 0, -1\}, \end{aligned}$$

where the last isomorphism is obtained by slicing the quotient by $z_2 = 1$. The stratum where $x_1 = 0$ (and similarly $x_2 = 0$) is isomorphic to a point (again by slicing):

$$\left[\frac{V(y_2^2 - 4x_2^3) \setminus V(x_2)}{\mathbb{G}_m} \right] \cong \operatorname{Spec} k. \qquad \square$$

Slicing lemma

Lemma 2. *Consider $\mathbb{A}^n = \operatorname{Spec} k[x_1, \ldots, x_n]$ with the $\mathbb{G}_m$-action for which $x_1, \ldots, x_n$ have weights $r_1, \ldots, r_n \geq 1$, and let $X \subset \mathbb{A}^n$ be a $\mathbb{G}_m$-invariant* *subscheme. Let* $f \in k[x_1, \ldots, x_n] \setminus \{0\}$ *be a homogeneous polynomial of degree* r *with respect to the* $(r_1, \ldots, r_n)$*-grading, and assume that* $X \subset \mathbb{A}^n \setminus V(f)$*. Then the natural map*

$$\left[\frac{X \cap V(f-1)}{\mu_r}\right] \to \left[\frac{X}{\mathbb{G}_m}\right]$$

is an isomorphism of stacks.