Notes on coalgebras

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1 Basic definitions

Let R be a (unital) commutative ring. All tensor products are taken over R.

Definition 1.1. A (coassociative counital) *coalgebra* over R is an R-module C with an R-linear map, the *coproduct*,

$$\Delta : C \longrightarrow C \otimes C$$

and an R-linear map, the *counit*,

$$\epsilon : C \longrightarrow R$$

such that the diagrams

$$\begin{array}{ccc} C & \xrightarrow{\ \Delta\ } & C \otimes C \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes\Delta} \\ C \otimes C & \xrightarrow[\Delta\otimes\mathrm{id}]{} & C \otimes C \otimes C \end{array}$$

and

$$\begin{array}{ccccc} & & C & & \\ & \overset{\sim}{\swarrow} & \downarrow{\scriptstyle \Delta} & \overset{\sim}{\searrow} & \\ C \otimes R & \xleftarrow[\mathrm{id}\otimes\epsilon]{} & C \otimes C & \xrightarrow[\epsilon\otimes\mathrm{id}]{} & R \otimes C \end{array}$$

commute. A morphism of (counital) coalgebras is an R-linear map that commutes with coproducts and preserves the counits.

Example 1. The trivial coalgebra is given by $C = R$ with coproduct $1 \longmapsto 1 \otimes 1$ and counit the identity.

It follows from coassociativity that we can define iterated coproducts $\Delta^n : C \longrightarrow C^{\otimes n+1}$.

If C is a coalgebra, then $C^\vee = \mathrm{Hom}_R(C, R)$ is an R-algebra, the multiplication being defined by

$$C^\vee \otimes C^\vee \longrightarrow (C \otimes C)^\vee \xrightarrow{\Delta^\vee} C^\vee.$$

Beware that the converse only holds under some finiteness assumption.

2 Cofree coalgebra

Definition 2.1. Let M be an R-module. The *cofree coalgebra* of M is a coalgebra CM over R with an R-linear map $CM \longrightarrow M$ such that for any coalgebra C over M, and any R-linear map $C \longrightarrow M$, there is a unique morphism $C \longrightarrow CM$ of coalgebras over R making the diagram

$$\begin{array}{ccc} & & CM \\ & \nearrow & \downarrow \\ C & \longrightarrow & M \end{array}$$

commute.

It follows from general categoric principles that the forgetful functor from the category of coalgebras over R to the category of R-modules admits a right adjoint (see [1] for the cocommutative case). In particular, the cofree coalgebra of an R-module always exist and is unique up to unique isomorphism.

We now describe an explicit construction of CM based in [2].

Definition 2.2. For any R-module M, the *tensor coalgebra* of M is the coalgebra TM with underlying R-module

$$TM = \bigoplus_{n \geq 0} T^n M, \qquad T^n M = M^{\otimes n};$$

its coproduct

$$\Delta : TM \longrightarrow TM \otimes TM = \bigoplus_{i,j \geq 0} T^i M \otimes T^j M$$

is given by the 'deconcatenation' map

$$x_1 \otimes \cdots \otimes x_n \longmapsto \sum_{i+j=n} (x_1 \otimes \cdots \otimes x_i) \otimes (x_{i+1} \otimes \cdots \otimes x_n)$$

and its counit

$$\epsilon : TM \longrightarrow R$$

is the projection onto $T^0 M = R$.

In general, the tensor coalgebra is *not* the cofree coalgebra.

Example 2. Let M be free of rank 1, so that we can identify the underlying R-module of TM with $R\langle X \rangle$ (polynomials in one variable) and that of $TM \otimes TM$ with $R\langle X_1, X_2 \rangle$ (polynomials in two non-commuting variables); under this identification the coproduct is given by

$$\Delta(X^n) = \sum_{i+j=n} X_1^i X_2^j \in R\langle X_1, X_2 \rangle.$$

If C is the trivial coalgebra, then the R-linear map $1 \longmapsto X$ from C to M cannot be lifted to a morphism of coalgebras $C \longrightarrow TM$. Indeed, any such lift would have to send 1 to an element $f = a_0 + a_1 X + \cdots + a_n X^n \in R\langle X \rangle$ satisfying $a_0 = 1$ (compatibility with counit), $\Delta(f) = f(X_1) f(X_2)$ (compatibility with coproduct), and $a_1 = 1$ (lifting). This is impossible.

The idea is to enlarge TM to a coalgebra where formulas such as $\Delta(f) = f \otimes f$ are also possible. In the example above, if we can consider power series and extend Δ to an R-linear map

$$\hat{\Delta} : R\langle\langle X \rangle\rangle \longrightarrow R\langle\langle X \rangle\rangle \hat{\otimes} R\langle\langle X \rangle\rangle = R\langle\langle X_1, X_2 \rangle\rangle$$

Now, the element

$$f = 1 + X + X^2 + X^3 + \cdots \in R\langle\langle X\rangle\rangle$$

is group-like:

$$\hat{\Delta}(f) = f(X_1)f(X_2).$$

The problem now is that $R\langle\langle X\rangle\rangle$ is 'too big' and $\hat{\Delta}$ is not a coproduct. We solve this by considering an appropriate submodule.

Definition 2.3. For an R-module M, the *completed tensor module* of M is the R-module

$$\hat{T}M = \prod_{n\geq 0} T^n M, \qquad T^n M = M^{\otimes n}$$

We also denote

$$\hat{T}M \hat{\otimes} \hat{T}M = \prod_{i,j\geq 0} T^i M \otimes T^j M.$$

The *completed deconcatenation map* is the R-linear map

$$\hat{\Delta} : \hat{T}M \longrightarrow \hat{T}M \hat{\otimes} \hat{T}M$$

defined by

$$(f_0, f_1, f_2, \ldots) \longmapsto \begin{pmatrix} f_0 & f_1 \otimes 1 & f_2 \otimes 1 & \cdots \\ 1 \otimes f_1 & f_1 \otimes f_1 & & \\ 1 \otimes f_2 & & \ddots & \\ \vdots & & & \end{pmatrix}.$$

Beware that the completed deconcatenation map is not a coproduct.

In what follows, we identify TM with the submodule of $\hat{T}M$ consisting of sequences $(f_n)_{n\geq 0}$ for which $f_n = 0$ for $n \gg 0$. Similarly for $TM \otimes TM$ inside $\hat{T}M \hat{\otimes} \hat{T}M$. Under these identifications, we have a commutative diagram

$$\begin{array}{ccc} TM & \xrightarrow{\Delta} & TM \otimes TM \\ \big\downarrow & & \big\downarrow \\ \hat{T}M & \xrightarrow[\hat{\Delta}]{} & \hat{T}M \hat{\otimes} \hat{T}M \end{array}$$

Remark 1. It is perhaps more intuitive to represent an element of $\hat{T}M$ as an infinite series

$$f = \sum_{n\geq 0} f_n, \qquad f_n \in T^n M.$$

The completed deconcatenation map is then given by

$$\hat{\Delta}(f) = \sum_{n\geq 0} \Delta(f_n).$$

Note that, for every submodule N of $\hat{T}M$ that contains TM, the natural map $N \otimes N \longrightarrow \hat{T}M \otimes \hat{T}M$ is injective (see [2] Lemma 3.28).

Theorem 2.4. *Let M be an R-module. Then CM can be identified with the largest submodule N of $\hat{T}M$ containing TM such that*

$$\hat{\Delta}(N) \subset N \otimes N.$$

Its coproduct is given by the restriction of $\hat{\Delta}$ and its counit is given by the projection on the 0th component.

Proof. Note that the sum of any two submodules N of $\hat{T}M$ containing TM such that $\hat{\Delta}(N) \subset N \otimes N$ still satisfies the same property. Thus, there exists a largest one. To prove that this is the cofree coalgebra, we verify the universal property. Given a coalgebra C, we can lift any R-linear map $\varphi : C \longrightarrow M$ to $\psi : C \longrightarrow \hat{T}M$ by setting

$$\psi(x) = (\epsilon(x), \varphi(x), \varphi^{\otimes 2}(\Delta^2(x)), \varphi^{\otimes 3}(\Delta^3(x)), \ldots).$$

With this definition, by writing $\Delta(x) = \sum_i y_i \otimes z_i$, one can check that

$$\hat{\Delta}(\psi(x)) = (\psi \otimes \psi)(\Delta(x)). \tag{1}$$

This proves that $\hat{\Delta}(\psi(C)) \subset \psi(C) \otimes \psi(C)$, so that $\psi(C) \subset N$, and that

$$\psi : C \longrightarrow N$$

is a morphism of coalgebras. The unicity follows easily from (1). □

3 Conilpotent coalgebras

Definition 3.1. A coalgebra C over R is *coaugmented* if there is given a morphism of coalgebras $u : R \longrightarrow C$.

The compatibility with counits implies that $\epsilon \circ u = \mathrm{id}_R$, so that

$$C = R \oplus \bar{C}, \qquad \bar{C} = \ker(\epsilon).$$

The *reduced coproduct* is the R-linear map

$$\bar{\Delta} : \bar{C} \longrightarrow \bar{C} \otimes \bar{C}$$

given by

$$\bar{\Delta}(x) = \Delta(x) - 1 \otimes x - x \otimes 1.$$

The iterations of $\bar{\Delta}$ are denoted by $\bar{\Delta}^n : \bar{C} \longrightarrow \bar{C}^{\otimes n+1}$.

Definition 3.2. The *coradical filtration* on a coaugmented coalgebra $C = R \oplus \bar{C}$ over R is the increasing filtration by submodules $F_0C \subset F_1C \subset \cdots$ defined by $F_0C = R$ and

$$F_nC = R \oplus \{x \in \bar{C} \mid \bar{\Delta}^r(x) = 0, \text{ for every } r \geq n\}.$$

Definition 3.3. A coalgebra C is *conilpotent* if it is coaugmented and if the coradical filtration is exhaustive:

$$C = \bigcup_{n \geq 0} F_nC.$$

References

[1] M. Barr, *Coalgebras over a commutative ring.* J. Algebra 32 (3) (1974) 600-610.

[2] M. Hazewinkel, *Cofree coalgebras and multivariable recursiveness.* J. Pure Appl. Algebra 183 (2003), no. 1-3, 61–103.

[3] J.-L. Loday, B. Vallette, *Algebraic Operads.* Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 346. Springer, Heidelberg, 2012.