

From multiple polylogarithms to the universal vector extension of an elliptic curve

Tiago J. Fonseca

IMECC - Unicamp, FAPESP

2021-2022 Fields Number Theory Seminar

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- ▶ Some algebro-geometric aspects on an **elliptic** analogue of the theory of **multiple polylogarithms** (Brown–Levin).

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- ▶ Classification of **unipotent connections** on punctured elliptic curves (Levin–Racinet, Hain, Enriquez–Etingof).
- ▶ Goal of the talk: make a case for the **universal vector extension** of an elliptic curve as the right framework to study these questions (Deligne).

► Polylogarithms:

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- ▶ Arithmetic phenomena when $z \in \overline{\mathbb{Q}}$.

Example (Special values of Dedekind zeta functions)

Zagier's conjecture:

$$\zeta_{\mathcal{F}}^*(1 - m) \sim \det(\mathcal{L}_m(\xi_j^\sigma)),$$

where $\mathcal{L}_m(z)$ are 'single-valued polylogarithms', e.g.

$$\mathcal{L}_2(z) = -2i\text{Im}(Li_2(z)) + 2 \log |z| \log(1 - \bar{z})$$

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Example (Multiple zeta values)

$$\zeta(k_1, \dots, k_r) = \sum_{n_1 > \dots > n_r > 0} \frac{1}{n_1^{k_1} \dots n_r^{k_r}}$$

Theorem (Brown '11): $\zeta(k_1, \dots, k_r)$ with $k_i \in \{2, 3\}$ span the \mathbb{Q} -vector space of MZVs.

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$$Li_2(z) = \int_0^z Li_1(y) \frac{dy}{y} = \int_0^z \int_0^y \frac{dx}{x-1} \frac{dy}{y}$$

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Define a connection $\nabla : \mathcal{O}_X^{\oplus 3} \rightarrow \Omega_X^1 \otimes \mathcal{O}_X^{\oplus 3}$ by

$$\nabla = d + A_0 \frac{dt}{t} + A_1 \frac{dt}{t-1}, \quad A_0 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

Fundamental system of solutions:

$$\begin{pmatrix} 1 & \log z & -Li_2(z) \\ 0 & 1 & -Li_1(z) \\ 0 & 0 & 1 \end{pmatrix}$$

Let X be a smooth variety over a field k .

- ▶ A **connection** on a vector bundle \mathcal{V} on X is a morphism of sheaves $\nabla : \mathcal{V} \rightarrow \Omega_{X/k}^1 \otimes \mathcal{V}$ satisfying

$$\nabla(fs) = df \otimes s + f\nabla(s)$$

where f (resp. s) is a section of \mathcal{O}_X (resp. \mathcal{V}). When $\mathcal{V} = \mathcal{O}_X \otimes V$, we can write

$$\nabla = d + A, \quad A \in \Gamma(X, \Omega_{X/k}^1 \otimes \text{End}_k(V)).$$

We say that ∇ is **integrable** (or **flat**) if $\nabla \circ \nabla = 0$, i.e.,

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- ▶ We say that (\mathcal{V}, ∇) is **unipotent** if there exists a filtration

$$0 = (\mathcal{V}_0, \nabla_0) \subset (\mathcal{V}_1, \nabla_1) \subset \cdots \subset (\mathcal{V}_n, \nabla_n) = (\mathcal{V}, \nabla)$$

such that

$$(\mathcal{V}_i, \nabla_i) / (\mathcal{V}_{i-1}, \nabla_{i-1}) \cong (\mathcal{O}_X, d).$$

Theorem

Let k be a field of characteristic zero. Every unipotent vector bundle with integrable connection on $X = \mathbb{P}_k^1 \setminus \{0, 1, \infty\}$ is *canonically* isomorphic to some

$$\left(\mathcal{O}_X \otimes V, d + \frac{dt}{t} \otimes A_0 + \frac{dt}{t-1} \otimes A_1\right),$$

where V is a finite dimensional k -vector space, and $A_0, A_1 \in \text{End}_k(V)$ are nilpotent.

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- ▶ Analytically, flat sections are given by MPLs.
- ▶ Related to KZ equations (Knizhnik–Zamolodchikov).
- ▶ $\pi_1^{\text{dR}}(X, x) \cong \text{Spec}(\bigoplus_{n \geq 0} (k \frac{dt}{t} \oplus k \frac{dt}{t-1})^{\otimes n})$ canonically.

Proof.

Let (\mathcal{V}, ∇) be a unipotent vector bundle with integrable connection on X . By unipotency, (\mathcal{V}, ∇) is regular singular at infinity and it extends canonically to a unipotent vector bundle with integrable logarithmic connection

$$\bar{\nabla} : \bar{\mathcal{V}} \rightarrow \Omega_{\mathbb{P}^1_k}^1(\log\{0, 1, \infty\}) \otimes \bar{\mathcal{V}}$$

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Since $\bar{\mathcal{V}}$ is unipotent and

$$H^0(\mathbb{P}_k^1, \mathcal{O}) = k, \quad H^1(\mathbb{P}_k^1, \mathcal{O}) = \text{Ext}^1(\mathcal{O}, \mathcal{O}) = 0$$

the canonical map $\mathcal{O} \otimes V \rightarrow \bar{\mathcal{V}}$ is an isomorphism, where $V = \Gamma(\mathbb{P}_k^1, \bar{\mathcal{V}})$.

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where $A \in \Gamma(\mathbb{P}_k^1, \Omega_{\mathbb{P}_k^1}^1(\log\{0, 1, \infty\})) \otimes \text{End}_k(V)$. To conclude, we remark that

$$\Gamma(\mathbb{P}_k^1, \Omega_{\mathbb{P}_k^1}^1(\log\{0, 1, \infty\})) = k \frac{dt}{t} \oplus k \frac{dt}{t-1}. \quad \square$$

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- ▶ Computes $L(E/\mathbb{Q}, 2)$ (Bloch, Beilinson, Goncharov–Levin).
- ▶ Elliptic polylogarithmic sheaves and the Eisenstein symbol (Beilinson, Levin, Deninger, etc.)

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- ▶ Let $r(z) = \text{Im}(z)/\text{Im}(\tau)$, and consider the 1-forms

$$\nu_{BL} = 2\pi i dr, \quad \omega_{BL}^{(n)}, \quad n \geq 0$$

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- ▶ MEPLs are iterated integrals of $\nu_{BL}, \omega_{BL}^{(n)}$ (agrees with q -averaging MPLs).

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- ▶ No such holomorphic f , but we can consider the **real-analytic** function r .

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- ▶ The **universal vector extension** of an elliptic curve $f : E \rightarrow S$ is a commutative group scheme $g : E^{\natural} \rightarrow S$ which sits into an exact sequence

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- ▶ $g : E^{\natural} \rightarrow S$ is a smooth group scheme of rel. dimension 2 (not proper neither affine!).

Theorem (Laumon '96)

If S is of characteristic zero, then $g_*\mathcal{O}_{E^{\natural}} \cong \mathcal{O}_S$ and $R^n g_*\mathcal{O}_{E^{\natural}} = 0$ for $n \geq 1$.

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Theorem (F.–Matthes '21)

There is a canonical decomposition

$$g_*\Omega_{E^{\natural}/S}^1(\log D) = g_*\Omega_{E^{\natural}/S}^1 \oplus \bigoplus_{n \geq 1} \mathcal{K}^{(n)}$$

where $\mathcal{K}^{(n)}$ are rank 1 subbundles uniquely determined by

1. $d\mathcal{K}^{(n)} = g_*\Omega_{E^{\natural}/S}^1 \wedge \mathcal{K}^{(n-1)}$, where $\mathcal{K}^{(0)} := f_*\Omega_{E/S}^1$,
2. $\mathcal{K}^{(n)} \wedge \mathcal{K}^{(0)} = 0$,
3. $\text{Res}_D(\mathcal{K}^{(n)})$ has degree $n - 1$.

If $\nu, \omega^{(0)}$ trivializes $g_* \Omega_{E^{\natural}/S}^1$, with $\omega^{(0)}$ in $\mathcal{K}^{(0)}$, then there are unique trivializations $\omega^{(n)}$ of $\mathcal{K}^{(n)}$ such that

1. $d\omega^{(n)} = \nu \wedge \omega^{(n-1)}$,
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Theorem (F.–Matthes '21)

Every relatively unipotent vector bundle with integrable connection on $E^{\natural} \setminus D$ over S is *canonically* isomorphic to some

$$(g^* \mathcal{W}, d + A \otimes \nu + B_0 \otimes \omega^{(0)} + \sum_{n \geq 1} ad_A^n(B_0) \otimes \omega^{(n)})$$

where \mathcal{W} is a vector bundle over S , and A, B_0 are nilpotent endomorphisms of \mathcal{W} .

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$$(g^* \mathcal{W}, d + A \otimes \nu + B_0 \otimes \omega^{(0)} + \sum_{n \geq 1} ad_A^n(B_0) \otimes \omega^{(n)})$$

where \mathcal{W} is a vector bundle over S , and A, B_0 are nilpotent endomorphisms of \mathcal{W} .

Note: pullback by $\pi : E^{\natural} \rightarrow E$ gives a classification of relatively unipotent vector bundles with integrable connection on $E \setminus \{O\}$.

If $\nu, \omega^{(0)}$ trivializes $g_* \Omega_{E^{\natural}/S}^1$, with $\omega^{(0)}$ in $\mathcal{K}^{(0)}$, then there are unique trivializations $\omega^{(n)}$ of $\mathcal{K}^{(n)}$ such that

1. $d\omega^{(n)} = \nu \wedge \omega^{(n-1)}$,
2. $\omega^{(n)} \wedge \omega^{(0)} = 0$,
3. $\text{Res}_D(\omega^{(n)}) = \frac{t^{n-1}}{(n-1)!}$, where $t : D \xrightarrow{\sim} \mathbb{A}_S^1$ is induced by ν .

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Then

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- ▶ Real-analytic flat sections of unipotent vector bundles with integrable connection on $E(\mathbb{C}) \setminus \{O\}$ are given by Brown–Levin’s MEPLs.

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9.2. **The formula.** The connection is defined by a 1-form

$$\omega \in \Omega^1(\mathbb{C} \times \mathfrak{h}, \log \Lambda) \otimes \text{End } \mathfrak{p}.$$

via the formula

$$\nabla f = df + \omega f$$

where $f : \mathbb{C} \times \mathfrak{h} \rightarrow \mathfrak{p}$ is a (locally defined) section of (9.1). Specifically,

$$\omega = \frac{1}{2\pi i} d\tau \otimes \mathbf{a} \frac{\partial}{\partial \mathbf{t}} + \psi + \nu$$

where

$$\psi = \sum_{m \geq 1} \left(\frac{(2\pi i)^{2m+1}}{(2m)!} G_{2m+2}(\tau) d\tau \otimes \sum_{\substack{j+k=2m+1 \\ j,k > 0}} (-1)^j [\text{ad}_{\mathbf{t}}^j(\mathbf{a}), \text{ad}_{\mathbf{t}}^k(\mathbf{a})] \frac{\partial}{\partial \mathbf{a}} \right)$$

and

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- ▶ Algebraicity results (Hain, Luo) rely on explicit formulas and well-chosen \mathbb{Q} -structures.

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Assume that S is a smooth k -scheme. Then

$$0 \rightarrow \Omega_{S/k}^1 \rightarrow g_* \Omega_{E^\natural/k}^1(\log D) \rightarrow g_* \Omega_{E^\natural/S}^1(\log D) \rightarrow 0$$

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- ▶ Example on the uniformization:

$$\tilde{\omega}^{(1)} = \left(\frac{\theta'_\tau(z)}{\theta_\tau(z)} + 2\pi i r \right) dz + \frac{1}{2\pi i} \left(\frac{1}{2} \frac{\theta''_\tau(z)}{\theta_\tau(z)} - \frac{1}{6} \frac{\theta'''_\tau(0)}{\theta'_\tau(0)} - \frac{(2\pi i r)^2}{2} \right) d\tau$$

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- ▶ Motivic theory (à la Brown) of elliptic multiple zeta values. Action of motivic Galois group. Explanation of modular/elliptic phenomena of MZVs.