

Heights II*

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1 Faltings height

Let M be a compact complex manifold of dimension n and let Ω_M^n be the sheaf of holomorphic n -forms. Since Ω_M^n is coherent, $\Gamma(M, \Omega_M^n)$ is a finite dimensional complex vector space.

Example 1.1. If M is a compact Riemann surface, then $\Gamma(M, \Omega_M^1)$ is g -dimensional, where g is the genus of M .

We remark that $\Gamma(M, \Omega_M^n)$ is a sub-vector space of $\Gamma(M, \mathcal{A}_M^n)$, the space of global C^∞ complex n -forms, and there is a natural complex conjugation on the later. We define

$$\begin{aligned} \langle , \rangle : \Gamma(M, \Omega_M^n) \times \Gamma(M, \Omega_M^n) &\rightarrow \mathbf{C} \\ (\omega, \eta) &\mapsto (-1)^{\frac{n(n-1)}{2}} \left(\frac{i}{2}\right)^n \int_M \omega \wedge \bar{\eta} \end{aligned}$$

Lemma 1.1. $\langle , \rangle$ is a hermitian product.

Proof. It is clear that $\langle , \rangle$ is linear on the first coordinate and anti-linear on the second one. For $\omega, \eta \in \Gamma(M, \Omega_M^n)$, we have

$$\overline{\langle \omega, \eta \rangle} = (-1)^n (-1)^{\frac{n(n-1)}{2}} \left(\frac{i}{2}\right)^n \int_M \bar{\omega} \wedge \eta = (-1)^n (-1)^{n^2} (-1)^{\frac{n(n-1)}{2}} \left(\frac{i}{2}\right)^n \int_M \eta \wedge \bar{\omega} = \langle \eta, \omega \rangle$$

Finally, we show that $\langle , \rangle$ is positive-definite. Let $\omega \in \Gamma(M, \Omega_M^n)$. Locally, we can write

$$\omega = f dz_1 \cdots dz_n$$

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where $f = f(z_1, \dots, z_n)$ is a holomorphic function. Thus, writing $z_j = x_j + iy_j$, we get

$$\begin{aligned}\omega \wedge \bar{\omega} &= |f|^2 dz_1 \wedge \dots \wedge dz_n \wedge \overline{dz_1} \wedge \dots \wedge \overline{dz_n} \\ &= (-1)^{\frac{n(n-1)}{2}} |f|^2 (dz_1 \wedge \overline{dz_1}) \wedge \dots \wedge (dz_n \wedge \overline{dz_n}) \\ &= (-1)^{\frac{n(n-1)}{2}} |f|^2 \left(\frac{2}{i}\right)^n (dx_1 \wedge dy_1) \wedge \dots \wedge (dx_n \wedge dy_n)\end{aligned}$$

and it is clear that $\langle \omega, \omega \rangle \geq 0$ and the equality holds if and only if $f = 0$ locally, i.e. $\omega = 0$. $\blacksquare$

Example 1.2. Let $M = \mathbf{C}/\Lambda$, where Λ is a lattice in $\mathbf{C}$, regarded as a real vector space. We let P be a fundamental domain for Λ . The sheaf Ω_M^1 is generated by the global section ω , whose pullback by $\mathbf{C} \rightarrow \mathbf{C}/\Lambda$ is $dz = dx + i dy$. We have:

$$\langle \omega, \omega \rangle = \frac{i}{2} \int_M \omega \wedge \bar{\omega} = \int_P dx \wedge dy$$

Thus $\langle \omega, \omega \rangle$ is the “area” of the torus M .

Now let K be a number field and A an abelian variety over K , of dimension g and with zero section $e : \text{Spec } K \rightarrow A$. If $\mathcal{A} \rightarrow \text{Spec } \mathcal{O}_K$ is the Néron model of A , consider the canonical sheaf $\Omega_{\mathcal{A}/\mathcal{O}_K}^g$. It is an invertible sheaf (and in general it is not trivial). If $s : \text{Spec } \mathcal{O}_K \rightarrow \mathcal{A}$ is the zero section, then we define the relative **dualizing sheaf**

$$\omega_A = s^* \Omega_{\mathcal{A}/\mathcal{O}_K}^g$$

It is an invertible sheaf on $\text{Spec } \mathcal{O}_K$, thus corresponds to a projective module of rank 1 over $\mathcal{O}_K$.

Remark 1.1. Notice that ω_A just depends on $\mathcal{A}^0$, the identity component of $\mathcal{A}$.

If $\sigma : K \rightarrow \mathbf{C}$ is an embedding, the commutativity of the diagram

$$\begin{array}{ccccc} A_\sigma & \longrightarrow & A & \longrightarrow & \mathcal{A} \\ e_\sigma \uparrow & & e \uparrow & & s \uparrow \\ \text{Spec } \mathbf{C} & \xrightarrow{\sigma} & \text{Spec } K & \longrightarrow & \text{Spec } \mathcal{O}_K \end{array}$$

shows that

$$\sigma^* \omega_A = \Gamma(\mathcal{A}_\sigma(\mathbf{C}), \Omega_{A_\sigma(\mathbf{C})}^n)$$

We consider the hermitian product h_σ on $\sigma^* \omega_A$ defined as above. These hermitian products are invariant under complex conjugation¹.

We have thus a hermitian line bundle $\bar{\omega}_A$ over $\mathcal{O}_K$.

Definition 1.1. The **Faltings height** of A is denoted and defined by

$$h_F(A) = \frac{1}{[K : \mathbf{Q}]} \widehat{\deg}(\bar{\omega}_A)$$

¹To see this, you must convince yourself that holomorphic forms on $A_{\bar{\sigma}}(\mathbf{C})$ correspond to anti-holomorphic forms on $A_\sigma(\mathbf{C})$ and vice-versa.

Remark 1.2. In general, $h_F(A)$ depends on K , i.e. $h_F(A) \neq h_F(A \otimes K')$ if $K' \supset K$ is a number field extension. However, if A is semi-stable (everywhere), then the formation of the identity component of the Néron model $\mathcal{A}^\circ$ commutes with extension of scalars ([1], Section 7.4, Corollary 4), and thus $h_F(A) = h_F(A \otimes K')$ for every number field extension $K' \supset K$, thanks to the $1/[K : \mathbf{Q}]$ factor on the definition. By the “semi-stable reduction theorem”, there exists a number field $L \supset K$ such that $A \otimes L$ is semi-stable, and we can define the **stable** (or **geometric**) Faltings height as $h_{F,\text{stable}}(A) = h_F(A \otimes L)$.

2 Elliptic curves

General references for this section are [6] and the Chapter 7 of [5].

2.1 Weierstraß models

Let k be any field of characteristic $\neq 2, 3$ and E an elliptic curve over k . By a classical argument using Riemann-Roch, one can prove that E is isomorphic to a projective plane curve of the form

$$E(g_2, g_3) : y^2 = 4x^3 - g_2x - g_3$$

satisfying $\Delta(g_2, g_3) = g_2^3 - 27g_3^2 \neq 0$. The elliptic curve $E(g_2, g_3)$ is a **Weierstraß elliptic curve** and if $E \cong E(g_2, g_3)$, we say that $E(g_2, g_3)$ is a **Weierstraß model** for E . The quantity $\Delta(g_2, g_3) \in k^\times$ is the **discriminant** of $E(g_2, g_3)$.

Weierstraß models are not unique, but we can describe all of them. Namely, one can prove that every automorphism of Weierstraß elliptic curves is induced by an automorphism of $\mathbf{P}_k^2$ given (on $\mathbf{A}_k^2 \subset \mathbf{P}_k^2$) by

$$(x, y) \mapsto (u^2x, u^3y)$$

for some $u \in k^\times$. In particular,

$$E(g_2, g_3) \cong E(g'_2, g'_3) \iff \exists u \in k^\times \text{ such that } g'_2 = u^4 g_2 \text{ and } g'_3 = u^6 g_3$$

Notice that, in this case, $\Delta(g'_2, g'_3) = u^{12} \Delta(g_2, g_3)$.

The last formula shows that, if $E \cong E(g_2, g_3)$,

$$j(E) = 1728 \frac{g_2^3}{\Delta(g_2, g_3)} \in k$$

does not depend on the chosen Weierstraß model. This is the **j -invariant** of E .

Recall the notations $\mathcal{M}_{g,d}$ and $M_{g,d}$ from “Heights I”. Notice that $\mathcal{M}_{1,1}(k)$ is just the set of all elliptic curves over k , up to isomorphism, because elliptic curves admit a “canonical” principal polarization, given by

$$\begin{aligned} E(k) &\rightarrow \text{Pic}^0(E) \\ P &\mapsto [P] - [O] \end{aligned}$$

where $O \in E(k)$ is the identity of $E(k)$.

Theorem 2.1. We have $M_{1,1} = \mathbf{A}_k^1$ and, for extensions $K \supset k$, the function $\mathcal{M}_{1,1}(K) \rightarrow M_{1,1}(K)$ is given by j . In particular, $j(E) = j(E')$ if and only if $E \otimes \bar{k} \cong E' \otimes \bar{k}$.

Yet another important invariant of a Weierstraß model $E \cong E(g_2, g_3)$ is its **Néron differential**

$$\pi(g_2, g_3) = \frac{dx}{y} \in \Gamma(E, \Omega_{E/k}^1) \setminus \{0\}$$

which is an invariant differential form.

One can even show that to fix a Weierstraß model of an elliptic curve is the same thing as fixing $\pi \in \Gamma(E, \Omega_{E/k}^1) \setminus \{0\}$ (see the formula below).

Notice that, for $u \in k^\times$, the change of coordinates $(x, y) \mapsto (u^2 x, u^3 y)$ induces the formula $\pi(u^4 g_2, u^6 g_3) = u^{-1} \pi(g_2, g_3)$. In particular, if we denote $\omega = \Gamma(E, \Omega_{E/k}^1)$, the form

$$s = \Delta(g_2, g_3) \pi(g_2, g_3)^{\otimes 12} \in \omega^{\otimes 12}$$

does not depend on the chosen Weierstraß model! For this reason, we denote it just by $s = \Delta \pi^{\otimes 12}$.

Remark 2.2. When the characteristic of k is 2 or 3, some elliptic curves do not admit a Weierstraß model as defined above, but there is a more general notion of Weierstraß model, namely

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6,$$

that encompasses all elliptic curves over a field of any characteristic. The major drawback is that the formulas get really ugly and complicated, so we stick to characteristics different to 2 and 3 to fix ideas.

Remark 2.3. It is worth to draw some comments on the relative case. A **relative elliptic curve** over S is an abelian scheme over S of relative dimension 1. One can define, more generally, a **relative curve of genus 1**, where singularities are admitted.

In this relative scenario, one can mimic the Riemann-Roch argument and define Weierstraß models, but now g_2 and g_3 are sections of $\mathcal{O}_S$. For $E \rightarrow S$ a relative curve of genus 1 with identity section $e : S \rightarrow E$, we denote $\omega = e^* \Omega_{E/S}^1$. One can also define discriminants, j -invariants and Néron differentials, which can be shown to be *generating global sections* π of ω . Additionally, one can show as before that the formation of

$$s = \Delta \pi^{\otimes 12} \in \Gamma(S, \omega^{12})$$

does not depend on the chosen Weierstraß model and commutes with base change on S . In other words $\Delta \pi^{\otimes 12}$ is a (geometric) *modular form* of weight 12. The interested reader can read Deligne's [3], where all of this is discussed.

2.2 Minimal Weierstraß models and reductions

The formula $\Delta(u^4 g_2, u^6 g_3) = u^{12} \Delta(g_2, g_3)$ also leads to the notion of a minimal Weierstraß model.

Let A be a discrete valuation ring, with fraction field K , residue field k and let $v : K \rightarrow \mathbb{Z} \cup \{\infty\}$ be its valuation. We will assume that the characteristic of k is different from 2 and 3 (otherwise, one has to work with the scary general Weierstraß models!)

Let E be an elliptic curve over K . By taking $u \in A$ with $v(u) \gg 0$, we can always find a Weierstraß model $E(g_2, g_3)$ for E with $g_2, g_3 \in A$. In this case, $v(\Delta(g_2, g_3)) \geq 0$ and one can search for such “integral” Weierstraß models that minimize this quantity.

Definition 2.1. A **minimal discriminant** Δ for E is some element $\Delta = \Delta(g_2, g_3) \in A$, for some Weierstraß model $E \cong E(g_2, g_3)$, such that

$$v(\Delta) = \min\{v(\Delta(g'_2, g'_3)) \mid E \cong E(g'_2, g'_3), g'_2, g'_3 \in A\}$$

In this case, we also say that $E(g_2, g_3)$ is a **minimal Weierstraß model** for E .²

Notice that minimal Weierstraß models always exist and two minimal discriminants differ only by an element of $A^\times$.

We shall use the following fact. A proof can be found in the first chapter of [1].

Proposition 2.1. Let E be an elliptic curve over K and let $\mathcal{E} \rightarrow \text{Spec} A$ be the Néron model of E . If $E \cong E(g_2, g_3)$ is a minimal Weierstraß model of E and

$$\mathcal{W} = \text{Proj} \frac{A[X, Y, Z]}{(Y^2Z - 4X^3 + g_2XZ^2 + g_3Z^3)}$$

then there is an isomorphism

$$\mathcal{E}^\circ \cong \mathcal{W}^{\text{sm}}$$

where $\mathcal{W}^{\text{sm}}$ is the smooth locus of $\mathcal{W} \rightarrow \text{Spec} A$.

For this reason, minimal discriminants and explicit equations can be used to study reductions of elliptic curves:

Proposition 2.2. Let Δ be a minimal discriminant for E and $j = j(E)$. Up to isomorphism of k -group-schemes, there are 3 possibilities for $\mathcal{E}^\circ \otimes k$:

1. We say that E has **good reduction** if $\mathcal{E}^\circ \otimes k$ is an elliptic curve. This happens if, and only if, $v(\Delta) = 0$.
2. We say that E has **multiplicative reduction** if $\mathcal{E}^\circ \otimes k \cong \mathbf{G}_{m,k}$. This happens if, and only if, $v(\Delta) > 0$ and $v(j) = -v(\Delta)$.
3. We say that E has **additive reduction** if $\mathcal{E}^\circ \otimes k \cong \mathbf{G}_{a,k}$. This happens if, and only if, $v(\Delta) > 0$ and $v(j) > -v(\Delta)$.

In cases 2 and 3, we also say that E has bad reduction.

For a proof, see [6]. It is clear by definition of semi-stability of an abelian variety that E is semi-stable if, and only if, E has good or multiplicative reduction.

If K is a number field and $\mathfrak{p}$ a maximal ideal of $\mathcal{O}_K$, we say that E is semi-stable at $\mathfrak{p}$ if E is semi-stable when regarded as an elliptic curve over $\text{Frac} \mathcal{O}_{K,\mathfrak{p}} = K$. We say that E is semi-stable (everywhere) if E is semi-stable at every maximal ideal.

2.3 Complex uniformization

Let $\mathfrak{h} = \{z \in \mathbb{C} \mid \Im z > 0\}$. Recall that to $\tau \in \mathfrak{h}$ we can associate a Weierstraß elliptic curve

$$E_\tau : y^2 = 4x^3 - g_2(\tau)x - g_3(\tau)$$

where $g_2, g_3 : \mathfrak{h} \rightarrow \mathbb{C}$ are holomorphic functions given by

$$g_2(\tau) = 60 \sum_{(n,m) \in \mathbb{Z}^2 \setminus \{0\}} \frac{1}{(n + m\tau)^4} \quad \text{and} \quad g_3(\tau) = 140 \sum_{(n,m) \in \mathbb{Z}^2 \setminus \{0\}} \frac{1}{(n + m\tau)^6}$$

²Beware that some (most?) people call “minimal Weierstraß model” the A -scheme $\mathcal{W}$ defined in the next proposition.

Definition 2.2. The functions $\Delta(\tau) = \Delta(g_2(\tau), g_3(\tau))$ and $j(\tau) = 1728g_2(\tau)^3/\Delta(\tau)$ are the **modular discriminant** and **modular invariant** respectively.

The important fact about the elliptic curves E_τ is that there is an isomorphism of complex manifolds

$$\begin{aligned}\varphi_\tau : \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau) &\rightarrow E_\tau(\mathbf{C}) \\ z &\mapsto (1 : \wp_\tau(z) : \wp'_\tau(z))\end{aligned}$$

such that $\varphi_\tau^*(dx/y) = dz$.

Now, consider the action of $\mathrm{SL}_2(\mathbf{Z})$ on $\mathfrak{h}$ given by

$$g \cdot \tau = \frac{a\tau + b}{c\tau + d}, \quad \text{where } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

It is well defined because

$$\Im(g \cdot \tau) = \frac{\Im \tau}{|c\tau + d|^2}$$

Theorem 2.4 (Uniformization). The function $j : \mathfrak{h} \rightarrow \mathbf{C}$ induces an isomorphism of complex manifolds $\mathfrak{h}/\mathrm{SL}_2(\mathbf{Z}) \rightarrow \mathbf{C}$. Equivalently, every complex elliptic curve is of the form E_τ for some $\tau \in \mathfrak{h}$.

The last fact we will need is that j and Δ are *modular* of weights 0 and 12 respectively. This means that, for every $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$, we have

$$\begin{aligned}j(\tau) &= j(g \cdot \tau) \\ \Delta(\tau) &= \frac{1}{(c\tau + d)^{12}} \Delta(g \cdot \tau)\end{aligned}$$

In particular, if $g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, we see that $j(\tau) = j(\tau + 1)$ and $\Delta(\tau) = \Delta(\tau + 1)$ for every $\tau \in \mathfrak{h}$. It follows that, if $q(\tau) = e^{2i\pi\tau}$, then Δ and j can be written as functions of q .

Theorem 2.5. We have:

$$j(\tau) = \frac{1}{q(\tau)} + 744 + \sum_{n \geq 1} c_n q(\tau)^n$$

for some $c_n \in \mathbf{N}$ and

$$\Delta(\tau) = (2\pi)^{12} q(\tau) \prod_{n=1}^{\infty} (1 - q(\tau)^n)^{24}$$

3 Comparison theorem for elliptic curves

Before comparing the Faltings height with the modular height, we need to find a more or less explicit formula for the Faltings height of an elliptic curve. The idea is that we can express it in terms of discriminants and then we can try to compare discriminants with j -invariants.

Theorem 3.1. Let K be a number field and E an elliptic curve over K . Then

$$h_F(E) = \frac{1}{12[K:\mathbf{Q}]} (\log |\mathcal{O}_K/\Delta_{E/K}| - \sum_{\sigma} \log (|\Delta(\tau_{\sigma})| (\Im \tau_{\sigma})^6))$$

where σ runs through the set of embeddings $\sigma : K \rightarrow \mathbf{C}$ and τ_{σ} are arbitrary elements of $\mathfrak{h}$ satisfying $E_{\tau_{\sigma}} \cong E_{\sigma}$.

We remark that the formula

$$\Im(g \cdot \tau) = \frac{\Im \tau}{|c\tau + d|^2}, \quad \text{where } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

shows that $\Delta(g \cdot \tau) (\Im g \cdot \tau)^6 = \Delta(\tau) (\Im \tau)^6$ for every $\tau \in \mathfrak{h}$ and $g \in \mathrm{SL}_2(\mathbf{Z})$. Thus the above expression for $h_F(E)$ is well-defined.

For the proof of the theorem, we shall need the following two lemmas.

Lemma 3.1. Let $\bar{L}$ be a hermitian line bundle over $\mathrm{Spec} \mathcal{O}_K$ and $n \in \mathbf{Z}$. Then

$$\widehat{\deg}(\bar{L}^{\otimes n}) = n \widehat{\deg}(\bar{L})$$

Proof. Easy. ■

Lemma 3.2. Let $\mathcal{E} \rightarrow \mathrm{Spec} \mathcal{O}_K$ be the Néron model of E and $e : \mathrm{Spec} \mathcal{O}_K \rightarrow \mathcal{E}$ its identity section. If $\omega = e^* \Omega_{\mathcal{E}/\mathcal{O}_K}^1$, then there exists a global section $s \in \Gamma(\mathrm{Spec} \mathcal{O}_K, \omega^{\otimes 12}) \setminus \{0\}$ such that:

1. For every maximal ideal $\mathfrak{p}$ of $\mathcal{O}_K$, $s_{\mathfrak{p}} = \Delta_{\mathfrak{p}} \pi_{\mathfrak{p}}^{\otimes 12}$, where $\Delta_{\mathfrak{p}}$ is a minimal discriminant of E at $\mathfrak{p}$ and $\pi_{\mathfrak{p}}$ is the Néron differential of a minimal Weierstraß model of E at $\mathfrak{p}$.³
2. For every embedding $\sigma : K \rightarrow \mathbf{C}$, $\sigma^* s = \Delta(\tau_{\sigma}) \pi_{\sigma}^{\otimes 12}$ where $\tau_{\sigma} \in \mathfrak{h}$ satisfies $E_{\sigma} \cong E_{\tau_{\sigma}}$ and π_{σ} is the Néron differential associated to $E_{\tau_{\sigma}}$.

Proof. We see ω as a $\mathcal{O}_K$ -module and $\omega \hookrightarrow \omega \otimes K$. Choose any Weierstraß model for E and consider $s = \Delta \pi^{\otimes 12} \in \omega \otimes K$, where Δ is the associated discriminant and π the Néron differential.

For a maximal ideal $\mathfrak{p}$ of $\mathcal{O}_K$, consider the localization $s_{\mathfrak{p}} \in \omega_{\mathfrak{p}} \otimes K$. Since “ $s = \Delta \pi^{\otimes 12}$ ” does not depend on the Weierstraß model (cf. argument after Theorem 2.1) we get $s_{\mathfrak{p}} = \Delta_{\mathfrak{p}} \pi_{\mathfrak{p}}^{\otimes 12}$. In particular, $s_{\mathfrak{p}} \in \omega_{\mathfrak{p}}$.⁴ This shows (1) and that $s \in \omega$. The point (2) follows by the same argument. ■

Proof of the theorem. Let ω and s be as in the preceding lemma. We will compute $\widehat{\deg}(\omega^{\otimes 12})$. For the “finite part”, we remark that

$$\omega^{\otimes 12}/\mathcal{O}_K s = \bigoplus_{\mathfrak{p}} \omega_{\mathfrak{p}}^{\otimes 12}/\mathcal{O}_{K,\mathfrak{p}} s_{\mathfrak{p}} = \bigoplus_{\mathfrak{p}} \mathcal{O}_{K,\mathfrak{p}} \pi_{\mathfrak{p}}^{\otimes 12}/\mathcal{O}_{K,\mathfrak{p}} \Delta_{\mathfrak{p}} \pi_{\mathfrak{p}}^{\otimes 12}$$

thus

$$\log |\omega^{\otimes 12}/\mathcal{O}_K s| = \sum_{\mathfrak{p}} \log |\mathcal{O}_{K,\mathfrak{p}}/\mathcal{O}_{K,\mathfrak{p}} \Delta_{\mathfrak{p}}| = \log |\mathcal{O}_K/\Delta_{E/K}|$$

³When we say “ E at $\mathfrak{p}$ ” we mean that K is seen as the fraction field of $\mathcal{O}_{K,\mathfrak{p}}$ and minimal Weierstraß models are taken with respect to the discrete valuation ring $\mathcal{O}_{K,\mathfrak{p}}$.

⁴We are using implicitly that the Néron model of E over $\mathcal{O}_{K,\mathfrak{p}}$ is just $\mathcal{E} \otimes \mathcal{O}_{K,\mathfrak{p}}$ and that the latter is given by a minimal Weierstraß model. See the first chapter of [1].

Now, to the infinite places. If $\sigma : K \rightarrow \mathbf{C}$ is any embedding, then

$$\|s\|_\sigma = |\Delta(\tau_\sigma)| \|\pi_\sigma^{\otimes 12}\|_\sigma = |\Delta(\tau_\sigma)| \|\pi_\sigma\|_\sigma^{12}$$

Now,

$$\|\pi_\sigma\|_\sigma^2 = \frac{i}{2} \int_{E_\sigma(\mathbf{C})} \pi_\sigma \wedge \bar{\pi}_\sigma = \frac{i}{2} \int_{\mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau_\sigma)} dz \wedge \overline{dz} = \Im \tau_\sigma$$

Thus

$$\begin{aligned} h_F(E) &= \frac{1}{[K:\mathbf{Q}]} \widehat{\deg \bar{\omega}} = \frac{1}{12[K:\mathbf{Q}]} \widehat{\deg \bar{\omega}^{\otimes 12}} \\ &= \frac{1}{12[K:\mathbf{Q}]} \left(\log |\mathcal{O}_K / \Delta_{E/K}| - \sum_\sigma \log (|\Delta(\tau_\sigma)| (\Im \tau_\sigma)^6) \right) \end{aligned}$$

■

Example 3.1. For $K = \mathbf{Q}$, let $\tau \in \mathfrak{h}$ be such that $E \cong E_\tau$. Then:

$$h_F(E) = \log |\Delta_{E/\mathbf{Q}}| - \log |\Delta(\tau)| - \frac{1}{2} \log |\Im \tau|$$

If τ can be chosen such that E_τ is a minimal Weierstraß model for E , then we get the simpler formula

$$h_F(E) = -\frac{1}{2} \log |\Im \tau|$$

but this is not always possible.⁵

Remark 3.2. We could have obtained the last formula more directly. If E is an elliptic curve over $\mathbf{Q}$ and ω is its dualizing sheaf, then there is some generator $\pi \in \omega$, since $\mathbf{Z}$ is a PID. By denoting $\pi_{\mathbf{C}}$ the associated holomorphic form on $E(\mathbf{C})$, we get:

$$h_F(E) = \log |\omega / \pi \mathbf{Z}| - \log \|\pi\|_\infty = -\frac{1}{2} \int_{E(\mathbf{C})} \pi_{\mathbf{C}} \wedge \bar{\pi}_{\mathbf{C}}$$

If there is $\tau \in \mathfrak{h}$ such that E_τ is a minimal Weierstraß model for E , then $\pi = \pm dx/y$ and thus $\pi_{\mathbf{C}} = \pm dz$ under the isomorphism $E \cong E_\tau \cong \mathbf{C}/(\mathbf{Z} + \tau \mathbf{Z})$. We then obtain the previous formula $h_F(E) = -\frac{1}{2} \log |\Im \tau|$.

Remark 3.3. The formula in the above example is so explicit that you can even compute some heights on the computer, since the software Sage can compute minimal discriminants and period lattices.

We will use the following notation:

$$\begin{aligned} \log^+ : \mathbf{R} &\rightarrow \mathbf{R} \\ t &\mapsto \log \max\{t, 1\} \end{aligned}$$

If E is an elliptic curve over a number field K , recall that the modular height of E is given by

$$h_M(E) = h(j(E)) = \sum_\nu \log^+ |j(E)|_\nu$$

where $h : \mathbf{P}^1(\overline{\mathbf{Q}}) \rightarrow \mathbf{R}$ is the Weil height of $\mathbf{P}^1$, ν runs through all places of K and $|\cdot|_\nu$ are normalized absolute values.

⁵Here's an example <http://math.stackexchange.com/questions/120401/form-of-minimal-integral-weierstrass-equation-for-elliptic-curve-over-q-with-g>.

Theorem 3.4 (Comparison theorem). For elliptic curves with semi-stable reduction over K , we have

$$h_F = \frac{1}{12} h_M + O(\log^+ h_M + 1)$$

Remark 3.5. Explicitly, this means that there is some constant $C > 0$ such that for every semi-stable elliptic curve over K , we have

$$\left| h_F(E) - \frac{1}{12} h_M(E) \right| \leq C \log^+ h_M(E) + C$$

In the previous theorem, we expressed the Faltings height of an elliptic curve in terms of discriminants. On the other hand, the modular height is expressed in terms of the j -invariant. The idea in the next proof is to “make the j -invariant” appear by comparing it with the discriminant.

Proof of the theorem. Let E be a semi-stable elliptic curve over K . Let $\Delta = \Delta_{E/K}$ and $j = j(E)$. First, we look to the finite part. Let S be the set of finite places (or, equivalently, maximal ideals) of K where E has bad reduction. Since E is semi-stable, if v is a finite place, then $v \in S \iff |j|_v > 1$. We get:

$$\begin{aligned} \log |\mathcal{O}_K / \Delta| &= \sum_{\mathfrak{p}} \log |\mathcal{O}_{K,\mathfrak{p}} / \mathfrak{p}^{\nu_{\mathfrak{p}}(\Delta)}| \\ &= \sum_{\mathfrak{p} \in S} \nu_{\mathfrak{p}}(\Delta) \log |\mathcal{O}_{K,\mathfrak{p}} / \mathfrak{p}| \\ &= - \sum_{\mathfrak{p} \in S} \nu_{\mathfrak{p}}(j) \log |\mathcal{O}_{K,\mathfrak{p}} / \mathfrak{p}| \\ &= [K : \mathbf{Q}] \sum_{v \in S} \log^+ |j|_v \\ &= [K : \mathbf{Q}] \sum_{v \nmid \infty} \log^+ |j|_v \end{aligned}$$

For the infinite part, chose $\tau_{\sigma} \in \mathfrak{h}$, for each embedding $\sigma : K \rightarrow \mathbf{C}$, such that $E_{\sigma} \cong E_{\tau_{\sigma}}$. Let us denote $\Delta_{\sigma} = \Delta(\tau_{\sigma})$ and $j_{\sigma} = j(\tau_{\sigma})$.

We remark first that we can always assume $\Im \tau_{\sigma} \geq \sqrt{3}/2$ (cf. in [5] the fundamental domain for the action of $SL_2(\mathbf{Z})$ on $\mathfrak{h}$). The main point is that, for $\tau \in \mathfrak{h}$ satisfying $\text{im } \tau \geq \sqrt{3}/2$, we have:

$$\log |\Delta(\tau)| = -\log^+ |j(\tau)| + O(1) \quad (*)$$

and

$$2\pi \Im \tau = \log^+ |j(\tau)| + O(1) \quad (**)$$

Let us finish the proof of the theorem assuming these estimates. By combining the last theorem, the last paragraph and $(*)$, we get

$$\begin{aligned} h_F(E) &= \frac{1}{12[K : \mathbf{Q}]} \left(\log |\mathcal{O}_K / \Delta| - \sum_{\sigma} \log |\Delta_{\sigma}| - 6 \sum_{\sigma} \log \Im \tau_{\sigma} \right) \\ &= \frac{1}{12[K : \mathbf{Q}]} \left([K : \mathbf{Q}] \sum_{v \nmid \infty} \log^+ |j|_v + \sum_{\sigma} \log^+ |j_{\sigma}| \right) - \frac{1}{2[K : \mathbf{Q}]} \sum_{\sigma} \log \Im \tau_{\sigma} + O(1) \\ &= \frac{1}{12} h_M(E) - \frac{1}{2[K : \mathbf{Q}]} \sum_{\sigma} \log \Im \tau_{\sigma} + O(1) \end{aligned}$$

Now, it is easy to prove that $\log^+ |j(\tau)| = \log \max\{|j(\tau)|, e\} + O(1)$ and we obtain by (**):

$$\log \Im \tau = \log \log \max\{|j(\tau)|, e\} + O(1)$$

Notice that $\log \log \max\{|j(\tau)|, e\} \geq 0$, thus

$$\begin{aligned} 0 &\leq \frac{1}{[K:\mathbf{Q}]} \sum_{\sigma} \log \log \max\{|j_{\sigma}|, e\} \\ &= \log \left(\prod_{\sigma} \log \max\{|j_{\sigma}|, e\} \right)^{\frac{1}{[K:\mathbf{Q}]}} \\ &\leq \log \left(\frac{1}{[K:\mathbf{Q}]} \sum_{\sigma} \log \max\{|j_{\sigma}|, e\} \right) \\ &\leq \log \left(\frac{1}{[K:\mathbf{Q}]} \sum_{\sigma} (\log^+ |j_{\sigma}| + 1) \right) \\ &\leq \log(h_M(E) + 1) \\ &= \log^+ h_M(E) + O(1) \end{aligned}$$

where the second inequality comes from the arithmetic-geometric mean inequality and the last equality comes from the fact that $\log(1+t) = \log^+ t + O(1)$ for $t \geq 0$.

To finish the proof, we must show why (*) and (**) hold. We look at q -expansions. Since $\Delta(\tau) = (2\pi)^{12} q(\tau) \prod_{n \geq 1} (1 - q(\tau)^n)^{24}$, we get

$$\log |\Delta(\tau)| = \log |q(\tau)| + \sum_{n \geq 1} 24 \log |1 - q(\tau)^n| + O(1)$$

Since $\text{im } \tau \geq \sqrt{3}/2$, we get $|q(\tau)| = e^{-2\pi \Im \tau} \leq e^{-\pi \sqrt{3}} < 1$. Thus,

$$0 \leq \sum_{n \geq 1} \log |1 - q(\tau)^n| \leq \sum_{n \geq 1} \log(1 + e^{-\pi \sqrt{3}n}) \leq \sum_{n \geq 1} e^{-\pi \sqrt{3}n} < \infty$$

and this shows that

$$\log |\Delta(\tau)| = \log |q(\tau)| + O(1)$$

Now, to prove (*), it is sufficient to prove that

$$\log^+ |j(\tau)| = -\log |q(\tau)| + O(1) \quad (***)$$

This comes in two steps. Recall that $q(\tau)j(\tau) = 1 + 744q(\tau) + \sum_{n \geq 1} c_n q(\tau)^{n+1}$.

1. The q -expansion of j shows that there exists $\epsilon > 0$ such that, for every τ satisfying $|j(\tau)| \leq 1$, we have $|q(\tau)| \geq \epsilon$. Hence, if $|j(\tau)| \leq 1$, $\log^+ |j(\tau)| + \log |q(\tau)| = \log |q(\tau)| = O(1)$.
2. If $|j(\tau)| \geq 1$, then $\log^+ |j(\tau)| + \log |q(\tau)| = \log |j(\tau)| + \log |q(\tau)| = \log |1 + 744q(\tau) + \sum_{n \geq 1} c_n q(\tau)^{n+1}| = O(1)$.

Finally, (**) is a direct consequence of the formula $\log |q(\tau)| = -2\pi \Im \tau$ and (***). ■

References

- [1] Bosch, S., Lütkebohmert, W., Raynaud, M., *Néron Models*. Springer-Verlag, 1990.
- [2] Chao Li, *Heights and Finiteness theorems of abelian varieties*. Available online at <http://www.math.harvard.edu/~chaoli/doc/HeightsAndFiniteness.html>.
- [3] Deligne, P., *Courbes elliptiques : formulaire d'après J. Tate mis au goût du jour*, in Modular Functions of One Variable IV, LNM 476, 1975.
- [4] Milne, J. S., *Abelian varieties*. Version 2.0. Available online at <http://www.jmilne.org/math/CourseNotes/AV.pdf>.
- [5] Serre, J-P., *Cours d'arithmétique*. 4e édition, PUF.
- [6] Silverman, J. H., *The Arithmetic of Elliptic Curves*, 2nd edition. GTM Springer-Verlag.