Notes on the hyperelliptic curve $y^2 = 1 - x^5$

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We work over $\overline{\mathbf{Q}}$. Let C be the hyperelliptic curve of genus 2 defined by $y^2 = 1 - x^5$. The chart at the point ∞ (corresponding to $s = t = 0$) is given by the equation $s^2 = t^6 - t = t(t^5 - 1)$ where $(x, y) = (1/t, s/t^3)$.

1 Complex multiplication by $\mathbf{Q}(\zeta_5)$

Every 5th root of unity $\zeta \in \overline{\mathbf{Q}}$ defines an automorphism of C by

$$\begin{aligned} \sigma_\zeta : C &\longrightarrow C \\ (x, y) &\longmapsto (\zeta x, y) \\ (t, s) &\longmapsto (\zeta^{-1} t, \zeta^{-3} s) \end{aligned}$$

This induces an action $\sigma : \mu_5 \longrightarrow \mathrm{Aut}(C)$. Note that each σ_ζ fixes ∞.

2 Real multiplication on the Jacobian

Let J be the Jacobian of C. If $f : C \longrightarrow J$ denotes the morphism given on points by

$$p \longmapsto [p] - [\infty],$$

then the image of f defines a divisor Θ on J. The morphism

$$\lambda : J \longrightarrow J^t$$

associated to the line bundle $\mathcal{O}_C(\Theta)$ is a principal polarization on J such that $\lambda^{-1} = -f^*$ (cf. [1] Lemma 6.9).

Let σ be an automorphism of C fixing ∞. Then $\sigma^* : J \longrightarrow J$ is an automorphism of abelian surfaces making the diagram

$$\begin{array}{ccc} C & \xrightarrow{\ \sigma\ } & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ J & \xleftarrow[\ \sigma^*\ ]{} & J \end{array}$$

commute. By duality and by the formula $\lambda^{-1} = -f^*$, we obtain the commutative diagram

$$\begin{array}{ccc} J & \xleftarrow{\ \sigma^*\ } & J \\ {\scriptstyle \lambda}\downarrow & & \downarrow{\scriptstyle \lambda} \\ J^t & \xrightarrow[\ \sigma^{*t}\ ]{} & J^t \end{array}$$

In particular, the group μ_5 acts on J by $\zeta \longmapsto \sigma_\zeta^* =: i(\zeta)$ and, if the Rosatti involution on $\mathrm{End}(J)$ associated to λ is denoted $\varphi^\dagger := \lambda^{-1} \circ \varphi^t \circ \lambda$, then

$$i(\zeta)^\dagger = i(\zeta)^{-1} = i(\zeta^{-1}).$$

Fix any primitive 5th root of unity ζ_5. Observe that i extends to a morphism of rings $i : \mathbf{Z}[\zeta_5] \longrightarrow \mathrm{End}(J)$. Let $\rho := \zeta_5 + \zeta_5^{-1}$ and set $R := \mathbf{Z}[\rho]$; this is the ring of integers of $\mathbf{Q}(\sqrt{5})$ (ρ may be identified with $(1-\sqrt{5})/2$). Now,

$$i(\rho)^\dagger = (i(\zeta_5) + i(\zeta_5^{-1}))^\dagger = i(\zeta_5^{-1}) + i(\zeta_5) = i(\rho).$$

As ρ generates R over $\mathbf{Z}$, we conclude that $i(r)^\dagger = i(r)$ for every $r \in R$. We have just proved that $(J, \lambda, i : R \longrightarrow \mathrm{End}(J))$ is a principally polarized abelian surface with real multiplication by R (or by $\mathbf{Q}(\sqrt{5})$).

3 Field of periods

Consider the rational 1-forms (defined over $\overline{\mathbf{Q}}$)

$$\omega_k := x^{k-1}\frac{dx}{y}$$

for $1 \leq k \leq 4$. It is easy to check that ω_1 is regular over $C \setminus \{\infty\}$; thus the same holds for all ω_k. Note that the rational function s is a uniformizer at ∞. In particular, x (resp. y) has a pole of order 2 (resp. 5) at ∞ and this implies that

1. ω_1 (resp. ω_2) is a form of the first kind (i.e. everywhere regular), with a zero of order 2 (resp. 0) at ∞, and
2. ω_3 (resp. ω_4) is a form of the second kind (i.e. all residues vanish), with a pole of order 2 (resp. 4) at ∞.

Each of the above forms define an element of $H_{dR} := H^1_{\mathrm{dR}}(C/\overline{\mathbf{Q}})$ and since they have distinct orders at ∞ they must be linearly independent. As $\dim H_{dR} = 4$, they must form a basis of this $\overline{\mathbf{Q}}$-vector space.

Consider the path

$$\begin{aligned} \epsilon : [0,1] &\longrightarrow C(\mathbf{C}) \\ u &\longmapsto (u, \sqrt{1-u^5}), \end{aligned}$$

denote by $\tau : C \longrightarrow C$ the hyperelliptic automorphism $(x,y) \longmapsto (x,-y)$, and let $\sigma := \sigma_{\zeta_5}$; note that τ commutes with σ. We may define a loop at $(0,1) \in C(\mathbf{C})$

$$c : [0,1] \longrightarrow C(\mathbf{C})$$

by

$$c := \epsilon \cdot (\tau \circ \epsilon)^{-1} \cdot (\sigma \circ \tau \circ \epsilon) \cdot (\sigma \circ \epsilon)^{-1},$$

where $\cdot$ denotes path composition and $^{-1}$ the operation on paths that reverses direction. The loop c defines a class γ in $H_B := H_1(C(\mathbf{C}), \mathbf{Q})$.

Recall that Euler's beta function is defined by

$$B(a,b) = \int_0^1 v^{a-1}(1-v)^{b-1}dv$$

for any $a, b \in \mathbf{C}$ with positive real part. By considering the change of variables $v = u^5$, $dv = 5u^4du = 5v^{4/5}du$, we obtain

$$\int_\epsilon \omega_k = \frac{1}{5}B\left(\frac{k}{5}, \frac{1}{2}\right).$$

As $\tau^*\omega_k = -\omega_k$ and $\sigma^*\omega_k = \zeta_5^k\omega_k$, we conclude that

$$\int_\gamma \omega_k = \int_\epsilon \omega_k - \int_\epsilon \tau^*\omega_k + \int_\epsilon (\sigma\circ\tau)^*\omega_k - \int_\epsilon \sigma^*\omega_k = \frac{2}{5}(1-\zeta_5^k)B\left(\frac{k}{5}, \frac{1}{2}\right).$$

Let σ_* denote the action of σ on H_B. We claim that the singular 1-chains $\gamma_1 := \gamma$, $\gamma_2 := \sigma_*\gamma$, $\gamma_3 := \sigma_*^2\gamma$, and $\gamma_4 := \sigma_*^3\gamma$ form a basis of the $\mathbf{Q}$-vector space H_B. Indeed, for every $1 \leq l \leq 4$, we have

$$\int_{\gamma_l} \omega_1 = \int_{\sigma_*^{l-1}\gamma} \omega_1 = \int_\gamma (\sigma^{l-1})^* \omega_1 = \zeta_5^{l-1} \int_\gamma \omega_1;$$

as $\int_\gamma \omega_1 \neq 0$ and $1, \zeta_5, \zeta_5^2, \zeta_5^3$ are $\mathbf{Q}$-linearly independent, we conclude that $\gamma_1, \ldots, \gamma_4$ must also be $\mathbf{Q}$-linearly independent.

Hence the field of periods $\mathcal{P}(C)$ of C is generated by the complex numbers

$$\int_{\gamma_l} \omega_k = \zeta_5^{k(l-1)} \int_\gamma \omega_k = \frac{2}{5} \zeta_5^{k(l-1)} (1 - \zeta_5^k) B\left(\frac{k}{5}, \frac{1}{2}\right),$$

that is,

$$\mathcal{P}(C) = \overline{\mathbf{Q}}(B(1/5, 1/2), B(2/5, 1/2), B(3/5, 1/2), B(4/5, 1/2)).$$

4 Special values of Γ and Grothendieck's period conjecture

We have

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

and the fundamental identities (cf. [3] XII 12.14)

$$\Gamma(1+a) = a\Gamma(a), \quad \Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin(\pi a)}, \quad B(a, 1/2) = 2^{2a-1} \frac{\Gamma(a)^2}{\Gamma(2a)}.$$

We thus obtain

$$\begin{aligned}
B\left(\frac{1}{5}, \frac{1}{2}\right) &= \frac{1}{2^{3/5}} \frac{\Gamma(1/5)^2}{\Gamma(2/5)} \\
B\left(\frac{2}{5}, \frac{1}{2}\right) &= \frac{\sin(\pi/5)}{2^{1/5}} \frac{\Gamma(1/5)\Gamma(2/5)^2}{\pi} \\
B\left(\frac{3}{5}, \frac{1}{2}\right) &= \frac{2^{1/5}}{5\sin(2\pi/5)} \frac{\pi^2}{\Gamma(2/5)^2\Gamma(1/5)} \\
B\left(\frac{4}{5}, \frac{1}{2}\right) &= \frac{2^{3/5} 3\sin(2\pi/5)}{5\sin(\pi/5)^2} \frac{\pi\Gamma(2/5)}{\Gamma(1/5)^2}
\end{aligned}$$

Then it is clear that we have a *finite* field extension

$$\mathcal{P}(C) \subset \overline{\mathbf{Q}}(\pi, \Gamma(1/5), \Gamma(2/5)).$$

Grothendieck's period conjecture predicts that

$$\mathrm{trdeg}_{\mathbf{Q}} \mathcal{P}(C) = \dim \mathrm{MT}(J),$$

where $MT(J)$ denotes the Mumford-Tate group of the Jacobian J of C. Since J has complex multiplication by the quartic CM-field not containing an imaginary quadratic subgield $\mathbf{Q}(\zeta_5)$, the Hodge group $\mathrm{Hg}(J)$ is the $\mathbf{Q}$-algebraic group $U_{\mathbf{Q}(\zeta_5)}$ given on R-algebras by

$$U_{\mathbf{Q}(\zeta_5)}(R) = \{x \in (\mathbf{Q}(\zeta_5) \otimes_{\mathbf{Q}} R)^\times \mid x\overline{x} = 1\}$$

where $x \longmapsto \overline{x}$ denotes the complex conjugation (cf. [2]). Its dimension is easily shown to be 2, so that $\dim \mathrm{MT}(J) = 1 + \dim \mathrm{Hg}(J) = 3$.

We conclude that Grothendieck's period conjecture applied to C predicts that π, $\Gamma(1/5)$, and $\Gamma(2/5)$ are algebraically independent over $\mathbf{Q}$. Currently, we only know that $\mathrm{trdeg}_{\mathbf{Q}} \overline{\mathbf{Q}}(\pi, \Gamma(1/5), \Gamma(2/5)) \geq 2$ (Chudnovsky-Vasil'ev).

5 Uniformization of J

Question: It follows from section 2 that there exists $\tau = (\tau_1, \tau_2) \in \mathbb{H}^2$ such that the abelian surface $\mathbf{C}^2/L$ where $L = \{(a_1\tau_1 + b_1, a_2\tau_2 + b_2) \in \mathbf{C}^2 \mid (a, b) \in \mathbf{Z}[\rho] \oplus \mathbf{Z}[\rho]^*\}$ is isomorphic to J. Can we compute τ explicitly?

6 Other references

- D. Mumford, *Tata Lectures on Theta II*, Vol. 43, Progress in Mathematics, Birkhäuser (1984).
- K. G. Vasil'ev, *Algebraic Independence of the Periods of Abelian Integrals*, Vol. 60, No. 5, Mathematical Notes (1996).
- M. Waldschmidt, *Nombres transcendants et groupes algébriques*, Vol. 69-70, Astérisque (1979).

References

[1] J.S. Milne, *Jacobian Varieties.* Chapter VII in "*Arithmetic Geometry*", Ed. G. Cornell and J.H. Silverman (1986), Springer-Verlag.

[2] B. J. J. Moonen, Y. G. Zarhin, *Hodge classes on abelian varieties of low dimension.* Preprint (1999) available at `arxiv:math/990113v2`.

[3] E. T. Whittaker, G. N. Watson, *A Course of Modern Analysis*, Fourth Edition (1927), Cambridge University Press.