A motivic approach to the Gross–Zagier conjecture

Tiago J. Fonseca

Unicamp

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- Joint work with **Francis Brown**.
- Preprint [arXiv:2508.04844]:
  Single-valued periods of meromorphic modular forms and a motivic interpretation of the Gross–Zagier conjecture
- Goals of the talk:
  - Explain the GZ conjecture;
  - Relate it to the moduli spaces $M_{1,n}$;
  - Discuss motivic aspects.
- Note: Li '23, Bruinier-Li-Yang '25: analytic proof.

The Gross–Zagier conjecture

- $\mathbb{H} = \{z \in \mathbb{C} : Im(z) > 0\}$ upper-half plane; $SL_2(\mathbb{Z})$-action:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

- Klein's $J : \mathbb{H} \to \mathbb{C}$ inducing $SL_2(\mathbb{Z})\backslash\mathbb{H} \xrightarrow{\sim} \mathbb{C}$.

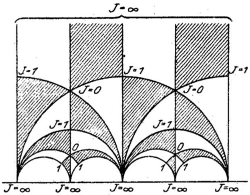


Fig. 7.

- Moduli: $SL_2(\mathbb{Z})\backslash\mathbb{H} \cong M_{1,1}$ via $z \mapsto (E_z = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}z), [0])$.

Theorem (Kronecker)

If $z \in \mathbb{H}$ is a quadratic number, then $J(z) \in \overline{\mathbb{Q}}$.

- Example (Weber):

$$J(\sqrt{-14}) = 6^{-3}\left(323 + 228\sqrt{2} + (231 + 161\sqrt{2})\sqrt{2\sqrt{2} - 1}\right)^3$$

- Geometric interpretation: an elliptic curve with too many endomorphisms (i.e., $End(E) \supsetneq \mathbb{Z}$) is defined over $\overline{\mathbb{Q}}$.
- We say $z \in \mathbb{H}$ is a CM point.

"The theory of complex multiplication, which forms a powerful link between number theory and analysis, is not only the most beautiful part of mathematics but also of all science." D. Hilbert

Gross–Zagier:

Invent. math. 84, 225–320 (1986)

Inventiones mathematicae

Heegner points and derivatives of *L*-series

Benedict H. Gross[1] and Don B. Zagier[2]

[1] Department of Mathematics, Brown University, Providence, RI 02912, USA
[2] Department of Mathematics, University of Maryland, College Park, MD 20742, USA and Max-Planck-Institut für Mathematik, Gottfried-Claren-Strasse 26, D-5300 Bonn 3, Federal Republic of Germany

- Key case of BSD conjecture.
- Computation involves higher Green's functions:

$$G_s(z, w) \in \mathbb{R}, \qquad z, w \in \mathbb{H}, \qquad s \geq 1$$

- 'Higher' versions of $G_1(z, w) = \log |J(z) - J(w)|^2$, but unclear geometric interpretation.

G_s are characterised by:

1. Symmetric real-valued real-analytic $SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z})$-invariant function on $\mathbb{H} \times \mathbb{H} \setminus \{(z, w) : z \notin SL_2(\mathbb{Z})\}$.
2. Eigenfunction for hyperbolic Laplacian in z and w:
$$\Delta_z G_s = s(s-1) G_s = \Delta_w G_s$$
3. Logarithmic singularities:
$$G_s(z, w) = \log|z - w|^r + O(1), \qquad z \to w$$
where $r = |Stab(w)|$ is almost always 2.
4. Vanishing at the cusps ($s > 1$):
$$G_s(z, w) = o(1), \qquad Im(z) \to \infty.$$

Explicit formula: infinite sum involving Legendre's function of second kind Q_{s-1}.

When $s \geq 1$ is an integer, G_s has remarkable arithmetic properties:

- GZ conjecture for $s \in \{1,2,3,4,5,7\}$: if z and w are CM of discriminants d_z and d_w, then

$$G_s(z,w) = (d_z d_w)^{\frac{s-1}{2}} \log|\alpha|, \qquad \text{for some } \alpha \in \overline{\mathbb{Q}}$$

  More complicated version for other s.
- For $G_1(z,w) = \log|J(z) - J(w)|^2$, it follows from Kronecker.
- Example (Mellit):

$$G_2(\tfrac{-1+i\sqrt{7}}{2}, i) = \tfrac{8}{\sqrt{7}} \log(8 - 3\sqrt{7}).$$

- Previous work: Gross–Kohnen–Zagier '87, Zhang '97, Mellit '08, Viazovska '11, ... , Li '23, Bruinier-Li-Yang '25.

Differential forms on $M_{1,n}$

Example (s=1)

- $G_1(z,w) = \log|J(z)-J(w)|^2$ as a definite integral:

$$G_1(z,w) = \int_1^{J(z)} \frac{dx}{x-J(w)} + \overline{\int_1^{J(z)} \frac{dx}{x-J(w)}}$$

- As a primitive:

$$d_z G_1(z,w) = \underbrace{\frac{J'(z)dz}{J(z)-J(w)}}_{\nu} + \overline{\frac{J'(z)dz}{J(z)-J(w)}} = \nu + \overline{\nu}$$

- ν is a '1-form of the 3rd kind' on $M_{1,1}$ with poles at $J(w)$.

Question: do $G_s(z,w)$ admit a similar description for $s>1$?

- $M_{1,n}$ = moduli of genus 1 curves with n marked points:

$\overline{M}_{1,n}$ = Deligne–Mumford compactification (stable curves).

- Action of S_n: permute the marked points.
- "Uniformisation":

$$\Gamma_{1,n} \backslash U_{1,n} \xrightarrow{\sim} M_{1,n}$$
$$(z, u_1, \ldots, u_{n-1}) \longmapsto (E_z = \mathbb{C}/(\mathbb{Z} + z\mathbb{Z}), [0], [u_1], \ldots, [u_{n-1}])$$

where

$$U_{1,n} = \{(z, u_1, \ldots, u_{n-1}) \in \mathbb{H} \times \mathbb{C}^{n-1} : u_i \notin \mathbb{Z} + \mathbb{Z}z,\ u_i - u_j \notin \mathbb{Z} + \mathbb{Z}z\}$$

$$\Gamma_{1,n} = SL_2(\mathbb{Z}) \ltimes (\mathbb{Z}^2)^{n-1}$$

- For $n \geq 3$ odd and $p+q=n-1,\ p,q\geq 0$, let

$$\Psi^{p,q}(z,w) = \sum_{\gamma \in SL_2(\mathbb{Z})} \frac{1}{(cz+d)^{n+1}} \frac{w-\overline{w}}{(\gamma z - w)^{p+1}(\gamma z - \overline{w})^{q+1}}$$

  and set

$$\nu^{p,q} = \Psi^{p,q}(z,w) dz \wedge du_1 \wedge \cdots \wedge du_{n-1}$$

- Let D_w be the fibre of $J(w) = (E_w, [0])$ under

$$\pi : \overline{M}_{1,n} \to \overline{M}_{1,1}, \qquad (C, p_0, \ldots, p_{n-1}) \mapsto (C, p_0)$$

Theorem (Brown–F.)

The above $\nu^{p,q}$ define algebraic n-forms on $\overline{M}_{1,n} \setminus D_w$ whose cohomology classes are alternating and satisfy

$$H^n(\overline{M}_{1,n} \setminus D_w)_{sgn} = H^n(\overline{M}_{1,n})_{sgn} \oplus \mathbb{C}[\nu^{0,n-1}] \oplus \cdots \oplus \mathbb{C}[\nu^{n-1,0}]$$

Let

$$\omega_i = du_i - \frac{u_i - \overline{u}_i}{z - \overline{z}} dz$$

and, for $r + s = n - 1$, set

$$\omega^r \overline{\omega}^s = \frac{1}{(n-1)!} \sum_{\sigma \in S_{n-1}} sgn(\sigma) \omega_{\sigma(1)} \wedge \cdots \wedge \omega_{\sigma(r)} \wedge \overline{\omega}_{\sigma(r+1)} \wedge \cdots \wedge \overline{\omega}_{\sigma(n-1)}$$

Theorem (Brown–F.)

There are unique real-analytic $(n-1)$-forms $g^{p,q}$ on $\overline{M}_{1,n} \setminus D_w$ such that

$$g^{p,q} = \sum_{r+s=n-1} G^{p,q}_{r,s}(z, w) \omega^r \overline{\omega}^s, \qquad dg^{p,q} = \nu^{p,q} + \overline{\nu^{q,p}}$$

Moreover,

$$G^{s-1,s-1}_{s-1,s-1}(z, w) = (-1)^{s-1} \binom{2s-2}{s-1} \frac{G_s(z, w)}{(z - \overline{z})^{s-1}(w - \overline{w})^{s-1}}$$

Upshot: given $n \geq 3$ odd,

- There's a whole $n \times n$ matrix of higher Green's functions.
- Entries are 'coefficients' of $(n-1)$-forms $g^{p,q}$ such that

$$dg^{p,q} = \nu^{p,q} + \overline{\nu^{q,p}}$$

  where $\nu^{p,q}$ are algebraic n-forms which split the Hodge filtration on $H^n_{dR}(\overline{M}_{1,n} \setminus D_w)_{sgn}$.
- If $n = 2s-1$, central entry is proportional to $G_s(z,w)$.
- Example:

$$\begin{pmatrix} G^{2,0}_{0,2}(z,w) & G^{1,0}_{0.2}(z,w) & G^{2,0}_{0,2}(z,w) \\ G^{2,0}_{1,1}(z,w) & G^{1,1}_{1,1}(z,w) & G^{0,2}_{1,1}(z,w) \\ G^{2,0}_{2,0}(z,w) & G^{1,1}_{0,2}(z,w) & G^{0,2}_{2,0}(z,w) \end{pmatrix}$$

  with $G^{1,1}_{1,1}(z,w) \sim G_2(z,w)$.

Motives

- Let $k \subset \mathbb{C}$ be a number field. Many cohomology theories for algebraic varieties over k, e.g.

Betti	Alg. de Rham	ℓ-adic
$H_B^\bullet(X)$	$H_{dR}^\bullet(X)$	$H_\ell^\bullet(X)$
$H_{sing}^\bullet(X(\mathbb{C});\mathbb{Q})$	$\mathbb{H}_{Zar}^\bullet(X;\Omega_{X/k}^\bullet)$	$\varprojlim_n H_{et}^\bullet(X;\mathbb{Z}/\ell^n)\otimes\mathbb{Q}_\ell$
$\mathbb{Q}$	k	$\mathbb{Q}_\ell$
complex conj.	Hodge filtration	Galois rep.

- Grothendieck comparison theorem (coefficients are periods):

$$H_{dR}^\bullet(X)\otimes_k \mathbb{C} \overset{\sim}{\to} H_B^\bullet(X)\otimes_{\mathbb{Q}}\mathbb{C}, \qquad [\omega]\mapsto \left([\sigma]\mapsto \int_\sigma \omega\right)$$

- Grothendieck: there should be a motive of X, namely some 'linear algebra object' (in some abelian category) containing all possible cohomological information of X, for any (Weil) cohomology theory.

- $DM(k)$ = Voevodsky's triangulated category of (geometric) motives over k.
- Morally: bounded derived category of the 'true' abelian category of motives over k.
- $DM(k)$ is $\mathbb{Q}$-linear, triangulated, pseudo-abelian, rigid tensor.
- Functor:

$$Y \subset X \qquad \longmapsto \qquad \underbrace{H(X,Y)}_{\text{"relative cohomology"}} \quad \in \quad DM(k)$$

  Denote $H(X,\emptyset) = H(X)$.
- Distinguished triangle:

$$H(Y)[-1] \longrightarrow H(X,Y) \longrightarrow H(X) \xrightarrow{+1}$$

  "relative cohomology long exact sequence"

- Realisation functors:

$$R_B : DM(k) \to D^b Vect_{\mathbb{Q}},\ R_{dR} : DM(k) \to D^b Vect_k,\ \ldots$$

  with $R_B H(X, Y) = H^\bullet_B(X, Y)$, ...
- Natural isomorphism $R_{dR} \otimes_k \mathbb{C} \cong R_B \otimes_{\mathbb{Q}} \mathbb{C}$.
- Basic motives:
  - Trivial: $\mathbb{Q}(0) = H(pt)$
  - Lefschetz: $\mathbb{Q}(-1) = H(\mathbb{A}^1 \setminus \{0\}, \{1\})[1]$.
  - Tate: $\mathbb{Q}(n) = \mathbb{Q}(-1)^{\otimes -n}$, $n \in \mathbb{Z}$.
- Tate twist: $M(n) = M \otimes \mathbb{Q}(n)$.
- Kummer motives:

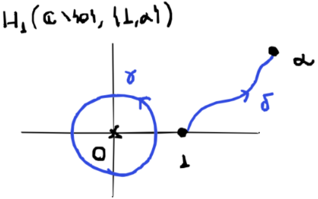
$$K_\alpha = H(\mathbb{A}^1 \setminus \{0\}, \{1, \alpha\})[1].$$

  Distinguished triangle:

$$\mathbb{Q}(0) \longrightarrow K_\alpha \longrightarrow \mathbb{Q}(-1) \xrightarrow{+1}$$

Logarithms of algebraic numbers are periods of Kummer motives:

- $R_B K_\alpha = H^1_B(\mathbb{A}^1 \setminus \{0\}, \{1, \alpha\})$ with $\mathbb{Q}$-basis $\delta^\vee, \gamma^\vee$.



- $R_{dR} K_\alpha = H^1_{dR}(\mathbb{A}^1 \setminus \{0\}, \{1, \alpha\})$ with k-basis $\frac{dz}{\alpha - 1}, \frac{dz}{z}$.
- Period matrix (representing comparison isomorphism):

$$P = \begin{pmatrix} \int_\delta \frac{dz}{\alpha-1} & \int_\delta \frac{dz}{z} \\ \int_\gamma \frac{dz}{\alpha-1} & \int_\gamma \frac{dz}{z} \end{pmatrix} = \begin{pmatrix} 1 & \log \alpha \\ 0 & 2\pi i \end{pmatrix}$$

- Single-valued period matrix:

$$S = \overline{P}^{-1} P = \begin{pmatrix} 1 & \log |\alpha|^2 \\ 0 & -1 \end{pmatrix}$$

Key point: Kummer motives generate every extension of $\mathbb{Q}(-1)$ by $\mathbb{Q}(0)$ in $DM(k)$ is given by Kummer motives, namely:

$$Ext^1_{DM(k)}(\mathbb{Q}(-1), \mathbb{Q}(0)) = k^\times \otimes_{\mathbb{Z}} \mathbb{Q}$$

Upshot:

To prove a number is a logarithm, write it as a period of some extension of $\mathbb{Q}(-1)$ by $\mathbb{Q}(0)$ in $DM(k)$.

Putting everything together

- Motives:

$$M_{n,z,w} = H(\overline{M}_{1,n} \setminus D_w, D_z)_{sgn}[n], \qquad F_{n,z} = H(D_z)_{sgn}[n-1]$$

- Biextension:

$$H(\overline{M}_{1,n})_{sgn}[n]$$
$$\downarrow$$
$$F_{n,z} \longrightarrow M_{n,z,w} \longrightarrow H(\overline{M}_{1,n} \setminus D_w)_{sgn}[n] \xrightarrow{+1}$$
$$\downarrow$$
$$F_{n,w}(-1)$$
$$\downarrow {+1}$$

- If $H(\overline{M}_{1,n})_{sgn}[n] = 0$, get distinguished triangle:

$$F_{n,z} \longrightarrow M_{n,z,w} \longrightarrow F_{n,w}(-1) \xrightarrow{+1}$$

Theorem (Brown-F.)

1. $G^{p,q}_{r,s}(z,w)$ *are single-valued periods of* $M_{n,z,w}$.
2. *If* z *is CM, then* $F_{n,z} = \mathbb{Q}(\frac{n-1}{2}) \oplus \widetilde{F}_{n,z}$ *after extending scalars.*
3. *If* z, w *are CM and* $H(\overline{M}_{1,n})_{sgn}[n] = 0$, *get subquotient*

$$\mathbb{Q}(\tfrac{n-1}{2}) \longrightarrow GZ_{n,z,w} \longrightarrow \mathbb{Q}(\tfrac{n-3}{2}) \xrightarrow{+1}$$

If $n = 2s-1$, *then* $G^{s-1,s-1}_{s-1,s-1}(z,w)$ *is 'the' single-valued period of* $GZ_{n,z,w}$. *The GZ conjecture holds for* $G_s(z,w)$.

We know that $R_B H(\overline{M}_{1,n})_{sgn}[n] = 0$ for $s \in \{1, 2, 3, 4, 5, 7\}$ (Faber–Consani + classical theory of modular forms).

Conjecture

The functor $R_B : DM(k) \to D^b Vect_{\mathbb{Q}}$ is conservative.

Known to hold on the subcategory of mixed Tate motives.

Theorem (Brown-F.)

The motive of $\overline{M}_{1,3}$ is mixed Tate. In particular, the GZ conjecture holds unconditionally for $n = 3$ (i.e., for $G_2(z, w)$).

We also expect the mixed Tate property for all $n \leq 10$.

Thank you!