Foliations in positive characteristic I

Tiago J. Fonseca

June 26, 2017

Contents

1 Foliations and the p^{th}-power map

Let X be a smooth algebraic (geometrically integral) variety of dimension n over a field k of characteristic $p > 0$. A *foliation* $\mathcal{F}$ on X is defined in the same way as a foliation in characteristic 0: it is the datum of an involutive coherent subsheaf $T\mathcal{F}$ of the tangent sheaf $TX := \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_{X/k}, \mathcal{O}_X) = \mathcal{D}er_k(\mathcal{O}_X)$ such that the quotient $TX/T\mathcal{F}$ is torsion free. The *singular locus* of $\mathrm{sing}(\mathcal{F})$ of $\mathcal{F}$ is the complement of the largest open subset of X over which $TX/T\mathcal{F}$ is locally free.

The main novelty in positive characteristic, in contrast with characteristic zero, is the presence of the p^{th}*-power map* on vector fields. For any vector field v defined on some open subset U of X, and any integer $n \geq 1$, we can regard v as a k-derivation of $\mathcal{O}_U$ and consider the k-linear endomorphism

$$v^n := \underbrace{v \circ \cdots \circ v}_{n \text{ times}}.$$

This is not a derivation in general. In fact, it follows from Leibiniz's rule that, for any regular functions f and g on U, we have

$$v^n(fg) = \sum_{j=0}^{n} \binom{n}{j} v^j(f) v^{n-j}(g).$$

Note however that, since k is of characteristic p, if we set $n = p$, then $\binom{p}{j} = 0$ for every $1 \leq j \leq p-1$. Thus

$$v^p(fg) = v^p(f)g + f v^p(g),$$

so that v^p *is* a derivation of $\mathcal{O}_U$, i.e. a vector field on U!

Example 1.1. Let $X = \mathbf{A}^2_k = \operatorname{Spec} k[x,y]$. Then it is easy to see that $\partial_x^p = \partial_y^p = 0$. It follows from Fermat's Little Theorem that $(x\partial_x)^p = x\partial_x$ (and same for y). For $p = 2$, one can easily compute

$$(y\partial_x + x\partial_y)^2 = x\partial_x + y\partial_y.$$

Giving a formula for the p^{th}-power of a general vector field can be quite complicated (cf. next section).

We come to our main definition.

Definition 1.2. We say that a foliation $\mathcal{F}$ on X is *p-closed* if $T\mathcal{F}$ is preserved under the p^{th}-power map, i.e. for any vector field v on some open subset of X, if v tangent to $\mathcal{F}$, then v^p is also tangent to $\mathcal{F}$.

The aim of this talk is to show that the presence — or the absence! — of p-closedness governs much of the global behavior of the foliation. Let us consider some baby examples.

Example 1.3. It follows from the last example that, if $p = 2$, then the foliation on $\mathbf{A}^2_k$ generated by the vector field $y\partial_x + x\partial_y$ is not p-closed. For general p, the foliation generated by $x\partial_x + ny\partial_y$ (n an integer) is p-closed.

We end this section with a basic lemma on the local form of a foliation that will be used throughout this text.

Lemma 1.4 (cf. [1] Lemma 6.1)**.** *Let $\mathcal{F}$ be a foliation of X of dimension r, and x be a point of $X \smallsetminus \operatorname{sing}(\mathcal{F})$. Then there exists a local system of coordinates $(s_1, \dots, s_n)$ of X at x and a trivialization $(v_1, \dots, v_r)$ of $T\mathcal{F}$ at x such that*

1. *$v_i(s_j) = \delta_{ij}$ for every $1 \leq i, j \leq r$;*
2. *$[v_i, v_j] = 0$ for every $1 \leq i, j \leq r$.*

If, moreover, $\mathcal{F}$ is p-closed, then

3. *$v_i^p = 0$ for every $1 \leq i \leq r$.*

Proof. Let $(s_1, \dots, s_n)$ be any local system of coordinates of X at x and $(\omega_{r+1}, \dots, \omega_n)$ be trivialization of $N^*\mathcal{F} = \mathcal{H}om_{\mathcal{O}_X}(TX/T\mathcal{F}, \mathcal{O}_X)$ at x. Since x is a non-singular point of $\mathcal{F}$, $N^*\mathcal{F}_x$ is a direct factor of $\Omega^1_{X,x}$, so that, up to renumbering $(s_1, \dots, s_n)$, we may assume that

$$(ds_1, \dots, ds_r, \omega_{r+1}, \dots, \omega_n)$$

is a basis of $\Omega^1_{X,x}$; let us set $\omega_i := ds_i$ for $1 \leq i \leq r$.

Let $(v_1, \dots, v_n)$ be the dual basis of $(\omega_1, \dots, \omega_n)$; (1) is clearly satisfied. Let $1 \leq i, j \leq r$; to prove that $[v_i, v_j] = 0$, it is sufficient to show that

$$\omega_k([v_i, v_j]) = 0$$

for every $1 \leq k \leq n$. Since $T\mathcal{F}$ is involutive, this is trivial for $r+1 \leq k \leq n$. For $1 \leq k \leq r$, we have

$$\omega_k([v_i, v_j]) = v_i(\omega_k(v_j)) - v_j(\omega_k(v_i)) + d\omega_k(v_i, v_j) = v_i(\delta_{kj}) - v_j(\delta_{ki}) - d^2 s_k(v_i, v_j) = 0.$$

This sets (2).

Finally let us prove (3). Assume that $\mathcal{F}$ is p-closed and fix $1 \leq i \leq r$. Then, there exist $f_1, \dots, f_r \in \mathcal{O}_{X,x}$ such that $v_i^p = \sum_{j=1}^r f_j v_j$. By (1), we have $f_j = v_i^p(s_j)$. Since $v_i(s_j) = \delta_{ij}$ is constant and $p \geq 2$, we get $f_j = 0$. ■

2 Quotients of p-closed foliations

We start with some purely algebraic preliminaries.

Definition 2.1. Let A be a ring of characteristic $p > 0$, and B be a subring containing the ring of p^{th}-powers A^p. We say that a tuple $(a_1, \ldots, a_r)$ of elements of A is a *p-basis* of A over B (resp. is *p-independent* over B) if the elements $a_1^{i_r} \cdots a_r^{i_r}$, with $1 \leq i_j \leq p-1$, form a basis of A as a B-module (resp. are B-linearly independent).

Remark 2.2. It is easy to see that, if A and B are integral domains and $(a_1, \ldots, a_r)$ is a p-basis of A over B, then it is also a p-basis of $\operatorname{Frac} A$ over $\operatorname{Frac} B$.

For fields, p-bases always exist, and admit a simple characterizations via differentials.

Proposition 2.3. *Let $K \supset L$ be a finite extension of fields of characteristic $p > 0$, and assume that L contains K^p. Then there exists a p-basis of K over L. Moreover, $(f_1, \ldots, f_r)$ is a p-basis of K over L if and only if it is a* differential basis *of K over L, i.e. $(df_1, \ldots, df_r)$ is a basis of the K-vector space $\Omega^1_{K/L}$.*

Proof. Let $(f_1, \ldots, f_m)$ be any tuple of p-independent elements of K over L. This is equivalent to $K' := K(f_1, \ldots, f_m)$ being an extension of L of degree p^m. If $f \in K \smallsetminus K'$, then the polynomial $T^p - f^p \in K'[T]$ is irreducible over K', so that $K'(f) = K(f_1, \ldots, f_m, f)$ is an extension of L degree p^{m+1}, i.e. $(f_1, \ldots, f_m, f)$ are p-independent. Since K is of finite degree over L, this shows that any family of p-independent elements of K can be completed to a p-basis of K over L.

If $(f_1, \ldots, f_r)$ is a p-basis of K over L, then it is easy to show that any function $\{f_1, \ldots, f_r\} \longrightarrow K$ lifts to an L-derivation of K. This shows that $(f_1, \ldots, f_r)$ is a differential basis of K over L. Conversely, assume that $f_1, \ldots, f_r \in K$ form a differential basis of K over L. If $f_1, \ldots, f_r$ are not p-independent, then we may assume that $f_r = P(f_1, \ldots, f_{r-1})$ for some $P \in L[T_1, \ldots, T_{r-1}]$. Thus $df_r = \sum_{j=1}^{r-1} (\partial_{T_j} P)(f_1, \ldots, f_{r-1}) df_j$, which contradics the linear independence of $df_1, \ldots, df_r$. By completing $(f_1, \ldots, f_r)$ to a p-basis $(f_1, \ldots, f_{r'})$, we see that both $(f_1, \ldots, f_{r'})$ ad $(f_1, \ldots, f_r)$ are differential bases of K over L, so that $r = r'$. ■

Corollary 2.4. *Let K be a function field over k. An element $f \in K$ satisfies $df = 0$ in $\Omega^1_{K/k}$ if and only if $f \in kK^p$ (compositum of k and K^p in K).*

Proof. Since K is a function field over k, K is a finite field extension of kK^p. Let $(f_1, \ldots, f_r)$ be a p-basis of K over kK^p, so that $(df_1, \ldots, df_r)$ is a basis of $\Omega^1_{K/kK^p} = \Omega^1_{K/k}$. If $f \in K$ satisfies $df = 0$, then, by writing f as a kK^p-linear combination of $f_1^{i_1} \cdots f_r^{i_r}$, $1 \leq i_j \leq p-1$, and by successively applying the derivatives ∂_{f_j} we see that $f \in kK^p$. ■

Remark 2.5. In the above corollary, if k is a perfect field, then $k \subset K^p$, so that $df = 0$ if and only if f is a p^{th}-power.

Let us come back to foliations. From now on, we shall assume that k is *perfect.* For any foliation $\mathcal{F}$ of X, we denote by $\mathcal{O}_X^{\mathcal{F}}$ the subsheaf of rings of $\mathcal{O}_X$ such that, for any affine open subset U of X,

$$\Gamma(U, \mathcal{O}_X^{\mathcal{F}}) = \{f \in \Gamma(U, \mathcal{O}_X) \mid v(f) = 0 \text{ for every } v \in \Gamma(U, T\mathcal{F})\}.$$

By the Leibniz's rule, it contains the subsheaf $\mathcal{O}_X^p$ consisting of p^{th}-powers of sections of $\mathcal{O}_X$.

Let Y be the scheme whose underlying topological space is X and whose sheaf of rings is $\mathcal{O}_X^{\mathcal{F}}$. Note that Y is an integral k-scheme. Denote by $\pi : X \longrightarrow Y$ the morphism of k-schemes given by the identity on the level of topological spaces, and induced by the inclusion $\mathcal{O}_X^{\mathcal{F}} \longrightarrow \mathcal{O}_X$ on the level of sheaves of rings.

Theorem. *Assume that $\mathcal{F}$ is a p-closed foliation of dimension r, and let $x \in X$ be a non-singular point of $\mathcal{F}$. Then there exists a system of local coordinates $(s_1, \ldots, s_n)$ of X at x such that*

1. *$(s_1, \ldots, s_r)$ is a p-basis of $\mathcal{O}_{X,x}$ over $\mathcal{O}_{Y,\pi(x)}$;*
2. *$(\partial_{s_1}, \ldots, \partial_{s_r})$ is a basis of $T\mathcal{F}_x$;*
3. *$\mathcal{O}_{Y,\pi(x)}$ is a regular local ring and $(s_1^p, \ldots, s_r^p, s_{r+1}, \ldots, s_n)$ form a system of local coordinates of Y at $\pi(x)$.*

The following proof was borrowed from [6].

Proof. Let $(t_1, \ldots, t_n)$ be a system of local coordinates of X at x and $(v_1, \ldots, v_r)$ be a trivialization of $T\mathcal{F}$ at x as in Lemma 1.4. Let us prove that $(t_1, \ldots, t_r)$ is a p-basis of $\mathcal{O}_{X,x}$ over $\mathcal{O}_{Y,\pi(x)}$.

It is easy to see that $(t_1, \ldots, t_r)$ is p-independent over $\mathcal{O}_{Y,\pi(x)}$. Thus, it suffices to prove that $\mathcal{O}_{Y,\pi(x)}[t_1, \ldots, t_r] = \mathcal{O}_{X,x}$. For any $1 \leq i \leq r+1$, consider the assertion

$$\text{There exists } f_i \in \mathcal{O}_{X,x} \smallsetminus \mathcal{O}_{Y,\pi(x)}[t_1, \ldots, t_r] \text{ such that } v_j(f_i) = 0 \text{ for every } 1 \leq j \leq i-1. \quad (A_i)$$

Let us prove that (A_i) implies (A_{i+1}). Since $v_i^p = 0$, there exists a smallest integer $0 \leq m \leq p-1$ such that $v_i^m(f_i) \notin \mathcal{O}_{Y,\pi(x)}[t_1, \ldots, t_r]$ but $v_i^{m+1}(f_i) \in \mathcal{O}_{Y,\pi(x)}[t_1, \ldots, t_r]$. Since $v_1, \ldots, v_r$ commute, we also have $v_j(v_i^m(f_i)) = 0$ for every $1 \leq j \leq i-1$. Thus, up to replacing f_i by $v_i^m(f_i)$, we may assume that $v_i(f_i) \in \mathcal{O}_{Y,\pi(x)}[t_1, \ldots, t_r]$. Since $v_j(v_i(f_i)) = 0$ for every $1 \leq j \leq i-1$, and $(t_1, \ldots, t_r)$ is a p-basis of $\mathcal{O}_{Y,\pi(x)}[t_1, \ldots, t_r]$ over $\mathcal{O}_{Y,\pi(x)}$, we can write $v_i(f_i)$ as a polynomial in $t_i, \ldots, t_r$ with coefficients in $\mathcal{O}_{Y,\pi(x)}$, with degree $\leq p-1$ in each t_j. Now, as $v_i^{p-1}(v_i(f_i)) = v_i^p(f_i) = 0$, the coefficient of t_i^{p-1} in this polynomial is zero, so that we can integrate it with respect to v_i. That is, there exists $h \in \mathcal{O}_{Y,\pi(x)}[t_1, \ldots, t_r]$ such that $v_i(h) = v_i(f_i)$. It is easy to see that $f_{i+1} := f_i - h$ satisfy the condition in (A_{i+1}).

Now, if $\mathcal{O}_{Y,\pi(x)}[t_1, \ldots, t_r] \neq \mathcal{O}_{X,x}$, then (A_1) holds. By induction, the above argument shows that (A_{r+1}) holds. This is impossible, since $v_j(f_{r+1}) = 0$ for every $1 \leq i \leq r$ implies that $f_{r+1} \in \mathcal{O}_{Y,\pi(x)}$, which contradicts $f_{r+1} \notin \mathcal{O}_{Y,\pi(x)}[t_1, \ldots, t_r]$. This finishes the proof that $(t_1, \ldots, t_r)$ is a p-basis of $\mathcal{O}_{X,x}$ over $\mathcal{O}_{Y,\pi(x)}$.

For every $r+1 \leq j \leq n$, write

$$t_j = g_{j,0} + \sum_{I \neq 0} g_{j,I} t_1^{i_1} \cdots t_r^{i_r},$$

where $I = (i_1, \ldots, i_r)$ are multi-indices satisfying $0 \leq i_k \leq p-1$, and $g_{j,I}$ are elements of $\mathcal{O}_{Y,\pi(x)}$; set $s_j := g_{j,0}$. For $1 \leq j \leq r$, set $s_j := t_j$. Then it is easy to prove that $(s_1, \ldots, s_n)$ is a system of local coordinates of X at x satisfying (2). Moreover, (1) was already established.

Finally, let us prove (3). Arguing as above, we see that $(s_1, \ldots, s_n)$ is a p-basis of $\mathcal{O}_{X,x}$ over $\mathcal{O}_{X,x}^p$. If $g \in \mathcal{O}_{Y,\pi(x)}$, let us write $g = P(s_1, \ldots, s_n)$ where P is a polynomial with coefficients in $\mathcal{O}_{X,x}^p$ of degree $\leq p-1$ in each variable. Since $\partial_{s_j} g = 0$ for every $1 \leq j \leq r$, we see that $g = P_0(s_{r+1}, \ldots, s_n)$, for some polynomial P_0 with coefficients in $\mathcal{O}_{X,x}^p$. This shows in particular that $\mathcal{O}_{Y,\pi(x)}$ is of finite type over the Noetherian ring $\mathcal{O}_{X,x}^P$ (because isomorphic to $\mathcal{O}_{X,x}$!), so that $\mathcal{O}_{Y,\pi(x)}$ is also Noetherian. It is clearly a local ring with maximal ideal $(s_1, \ldots, s_n) \cap \mathcal{O}_{Y,\pi(x)}$. Now, if g as above is in the maximal ideal of $\mathcal{O}_{Y,\pi(x)}$, then its constant term lies in $(s_1, \ldots, s_n) \cap \mathcal{O}_{X,x}^p = (s_1^p, \ldots, s_n^p)$. This proves that $(s_1^p, \ldots, s_r^p, s_{r+1}, \ldots, s_n)$ is a system of generators of the maximal ideal of $\mathcal{O}_{Y,\pi(x)}$. Since $\mathcal{O}_{X,x}$ is integral over $\mathcal{O}_{Y,\pi(x)}$, we have $\dim \mathcal{O}_{Y,\pi(x)} = \dim \mathcal{O}_{X,x} = n$, so that $\mathcal{O}_{Y,\pi(x)}$ is indeed regular. ■

The above theorem shows that Y can be thought as the *quotient* of X by the foliation $\mathcal{F}$. More precisely, we have the following result.

Corollary 2.6. *The coherent subsheaf $T\mathcal{F}$ of TX coincides with the kernel of $D\pi : TX \longrightarrow \pi^*TY$ over the open subset $X \smallsetminus \operatorname{sing}(\mathcal{F})$.*

Proof. By definition of Y, it is clear that $T\mathcal{F}$ is contained in the kernel of $D\pi$. Conversely, let x be a point of $X \smallsetminus \operatorname{sing}(\mathcal{F})$ and take a local system of coordinates $(s_1, \ldots, v_n)$ of X at x as in the theorem. For any $v \in TX_x$, we can write $v = \sum_{1\leq i\leq n} f_i \partial_{s_i}$. If $v \in \ker D\pi_x$, then, since $(s_1^p, \ldots, s_r^p, s_{r+1}, \ldots, s_n)$ form a system of local coordinates of Y at $\pi(x)$, we have in particular $0 = v(s_i) = f_i$ for every $r+1 \leq i \leq n$. Thus $v \in T\mathcal{F}_x = \bigoplus_{i=1}^r \mathcal{O}_{X,x}\partial_{s_i}$. ■

Corollary 2.7. *Let $\mathcal{F}$ be a foliation of X of codimension one. Then $\mathcal{F}$ is p-closed if and only if $\mathcal{F}$ admits a first integral.*

Recall that, if $\omega \in \Gamma(X, N\mathcal{F} \otimes \Omega^1_{X/k})$ is the 1-form defining $\mathcal{F}$, a *first integral* of $\mathcal{F}$ is a rational function f on X such that $df \neq 0$ and $\omega \wedge df = 0$.

Proof. Assume that the foliation $\mathcal{F}$ admits a first integral f and let v be a vector field tangent to $\mathcal{F}$, i.e. $\omega(v) = 0$. Since $\omega \wedge df = 0$, we have $v(f) = 0$, so that $v^p(f) = 0$. Furthermore, as $df \neq 0$ and

$$0 = i_{v^p}(\omega \wedge df) = \omega(v^p)df - v^p(f)\omega = \omega(v^p)df,$$

we conclude that $\omega(v^p) = 0$. This shows that $\mathcal{F}$ is p-closed.

Conversely, if $\mathcal{F}$ is p-closed, then there is a quotient $\pi : X \longrightarrow Y$ of X by $\mathcal{F}$. Let g be a rational function on Y and set $f := \pi^*g$. Since $T\mathcal{F}$ coincides with the kernel of $D\pi$ over $X \smallsetminus \operatorname{sing}(\mathcal{F})$, it is clear that $\omega \wedge df = 0$. To finish the proof, it is sufficient to remark that we can always choose g such that $df \neq 0$[1]. ■

3 The formulas of Jacobson and Hochschild

In the next section, we shall need to understand the behavior of the p^{th}-power map on vector fields of X under the $\mathcal{O}_X$-module structure of the tangent bundle TX.

Proposition 3.1. *Let v and w be vector fields on some open subset of X and f be a regular function defined on this same open subset. We have*

1. *Jacobson's formula:*

$$(v+w)^p = v^p + w^p + \sum_{i=1}^{p-1} s_i(v,w)$$

where $is_i(v,w)$ is the coefficient of T^{i-1} in

$$\underbrace{[Tv+w, \ldots [Tv+w, [Tv+w,}_{p-1 \text{ times}} v]] \ldots]$$

2. *Hochschild's formula:*

$$(fv)^p = f^p v^p + (fv)^{p-1}(f)v.$$

---

[1] Otherwise, we would have $k(Y) = k(X)^p$ (cf. [ref]), which contradicts [ref].

These are purely algebraic statements and hold in much greater generality. We prove them in the appendix.

Remark 3.2 (cf. [3] p. 187)**.** One can easily compute $s_i(v,w)$ by hand for small values of p:

p	$is_i(v,w)$
2	$s_1(v,w) = [v,w]$
3	$s_1(v,w) = [[v,w],w]$ $2s_2(v,w) = [[v,w],v]$
5	$s_1(v,w) = [[[[v,w],w],w],w]$ $2s_2(v,w) = [[[[v,w],v],w],w] + [[[[v,w],w],v],w] + [[[[v,w],w],w],v]$ $3s_2(v,w) = [[[[v,w],v],v],v] + [[[[v,w],v],w],v] + [[[[v,w],w],v],v]$ $4s_4(v,w) = [[[[v,w],v],v],v]$
$\vdots$	$\vdots$

To our purposes, the following corollary of these formulas is enough.

Corollary 3.3. *Let $\mathcal{F}$ be a foliation in X and $U \subset X$ an open subset. If $v_1, \dots, v_k \in \Gamma(U, T\mathcal{F})$ and $f_1, \dots, f_k \in \Gamma(U, \mathcal{O}_X)$, then the vector field on U*

$$(f_1 v_1 + \cdots + f_k v_k)^p - f_1^p v_1^p - \cdots - f_k^p v_k^p$$

is tangent to $\mathcal{F}$.

Clearly, an analogous statement is true for rational vector fields v_i and rational functions f_i on X.

4 Integrating factors of non p-closed foliations

We have seen that a codimension one foliation $\mathcal{F}$ of a smooth variety X over a field k of characteristic p admits a first integral if and only if it is p-closed. What can be said if $\mathcal{F}$ is not p-closed?

Proposition 4.1. *Let X be a smooth variety over a field k of characteristic $p > 0$, and $\mathcal{F}$ be a foliation on X of codimension one. If $\mathcal{F}$ is not p-closed, then any rational function $f \neq 0$ on X of the form $f = \omega_{\mathcal{F}}(v^p)$, for some rational vector field v tangent to $\mathcal{F}$, is an integrating factor of $\mathcal{F}$. Moreover, the ratio between any two such integrating factors f, g is the p-power of a rational function on X, i.e. $f = h^p g$ for some rational function h on X; in particular $\operatorname{dlog} f = \operatorname{dlog} g$.*

Proof. Let $v_1, \dots, v_{n-1}$ be commuting rational vector fields on X forming a basis of $T\mathcal{F} \otimes R_X$ (see Lemma 1.4), and let $f_1, \dots, f_{n-1} \in R_X$ be rational functions such that $v = f_1 v_1 + \cdots + f_{n-1} v_{n-1}$. Consider the rational vector field

$$v_n := f_1^p v_1^p + \cdots + f_{n-1}^p v_{n-1}^p.$$

It follows from Lemma 3.3 that $v^p - v_n \in T\mathcal{F} \otimes R_X$, so that

$$f = \omega_{\mathcal{F}}(v^p) = \omega_{\mathcal{F}}(v_n).$$

Furthermore, for any $1 \leq m \leq n-1$,

$$[v_m, v_n] = \sum_{i=1}^{n-1} [v_m, f_i^p v_i^p] \overset{(*)}{=} \sum_{i=1}^{n-1} f_i^p [v_m, v_i^p] \overset{(**)}{=} 0,$$

where in $(*)$ we used that the derivative of any p-power is zero in characteristic p, and in $(**)$ we used the formula in Remark A.2 and the fact that $v_1, \dots, v_{n-1}$ commute.

Let $\eta := f^{-1}\omega_{\mathcal{F}}$; we must prove that η is closed. Since $\omega_{\mathcal{F}}$ is completely integrable, the same holds for η, that is, $\eta \wedge d\eta = 0$. Thus

$$0 = i_{v_n}(\eta \wedge d\eta) = \eta(v_n)d\eta - \eta \wedge i_{v_n}(d\eta) = d\eta - \eta \wedge i_{v_n}(d\eta).$$

Therefore, it suffices to prove that $i_{v_n}(d\eta) = 0$. Since $(v_1, \dots, v_n)$ is a basis of $TX \otimes R_X$, it is enough to check that $i_{v_n}(d\eta)(v_m) = 0$ for any $1 \leq m \leq n$. This follows from the formula

$$\eta([v_n, v_m]) = v_n(\eta(v_m)) - v_m(\eta(v_n)) - d\eta(v_n, v_m) = v_n(\eta(v_k)) - v_k(\eta(v_n)) - i_{v_n}(d\eta)(v_m),$$

and from the fact that $v_1, \dots, v_n$ are commuting vector fields for which $\eta(v_m)$ is a constant for every $1 \leq m \leq n$. ■

A Proof of Jacobson's and Hochschild's formulas

A.1 Jacobson's formula

Let R be an associative ring, not necessarily commutative (e.g. R the ring of linear endomorphisms $\operatorname{End}(A)$ of a commutative ring A). For $x \in R$, we denote by l_x (resp. r_x) the endomorphism of R given by left (resp. right) multiplication by x, and we set $s_x := l_x - r_x$. Thus

$$s_x(y) = [x, y] = xy - yx$$

for any $x, y \in R$.

Proposition A.1 (Jacobson, cf. [3] V.63)**.** *Let R be an associative ring of characteristic $p > 0$. Then, for every $x, y \in R$, we have*

$$(x+y)^p = x^p + y^p + \sum_{i=1}^{p-1} s_i(x, y) \in R$$

where $is_i(x, y)$ is the coefficient of T^{i-1} in the polynomial $s_{Tx+y}^{p-1}(x) \in R[T]$.

Proof. Let $x \in R$. Since R is of characteristic p, and the endomorphisms l_x and r_x commute, we have

$$s_x^p = (l_x - r_x)^p = l_x^p - r_x^p$$

By factoring $l_x^p - r_x^p = (l_x - r_x)(l_x^{p-1} + l_x^{p-2}r_x + \dots + r_x^{p-1})$, we also obtain

$$s_x^{p-1} = \sum_{i=0}^{p-1} l_x^{p-1-i} r_x^i. \tag{A.1}$$

In the ring $R[T]$, we may write

$$(xT + y)^p = x^p T^p + y^p + \sum_{i=1}^{p-1} s'_i(x, y) T^i \tag{A.2}$$

for some $s'_i(x,y) \in R$. By differentiating the above identity with respect to the variable T, we obtain (recall that R is not necessarily commutative!)

$$\sum_{i=0}^{p-1}(Tx+y)^i x(Tx+y)^{p-1-i} = \sum_{i=1}^{p-1} i s'_i(x,y)T^{i-1}.$$

Now, by (A.1), the left-hand side in the above identity is $s_{Tx+y}^{p-1}(x)$. By setting $s_i(x,y) := s'_i(x,y)$, and by specializing (A.2) at $T=1$, we obtain the desired identity. ■

Remark A.2. The above formula $s_x^p = l_x^p - r_x^p$ means that, for any $x, y \in R$,

$$\underbrace{[x, \ldots, [x[x,y]]}_{p \text{ times}} \ldots] = [x^p, y].$$

A.2 Hochschild's formula

Our exposition is based on [5] Chapter 9, Paragraph 25.

In what follows, if A is any ring and $a \in A$, we still denote by a the element of $\mathrm{End}(A)$ given by left multiplication by a. In particular, if $D \in \mathrm{Der}(A)$, and $k \geq 0$ is an integer, Leibniz formula gives

$$D^k a := D^k \circ a = \sum_{i=0}^{k} \binom{k}{i} D^{k-i}(a) D^i \in \mathrm{End}(A).$$

Proposition A.3. *Let K be a field of characteristic $p > 0$, and D be a non-trivial derivation of K. Let $a_0, \ldots, a_{p-1} \in K$. If the linear endomorphism of K*

$$a_0 + a_1 D + \cdots + a_{p-1} D^{p-1}$$

is a derivation, then $a_0 = a_2 = \cdots = a_{p-1} = 0$.

Proof. We first prove that $1, D, \ldots, D^{p-1}$ are linearly independent over K. By contradiction, assume that there exists an integer $0 \leq i < p$ such that $1, D, \ldots, D^{i-1}$ are linearly independent over K, but $1, D, \ldots, D^i$ are not. Then there exist $b_0, \ldots, b_{i-1}$ such that

$$D^i = b_0 + b_1 D + \cdots + b_{i-1} D^{i-1}.$$

Let $a \in A$ be such that $D(a) \neq 0$. We compose the above identity on the right with left multiplication by a, and apply Leibniz formula:

$$\sum_{j=0}^{i} \binom{i}{j} D^{i-j}(a) D^j = b_0 a + b_1 \sum_{j=0}^{1} \binom{1}{j} D^{1-j}(a) D^j + \cdots + b_{i-1} \sum_{j=0}^{i-1} \binom{i-1}{j} D^{i-1-j}(a) D^j$$

so that

$$a D^i + i D(a) D^{i-1} - b_{i-1} a D^{i-1} \in \langle 1, D, \ldots, D^{i-2} \rangle.$$

But our original relation implies that $aD^i - b_{i-1} a D^{i-1}$ is also in $\langle 1, D, \ldots, D^{i-2} \rangle$. Since $iD(a) \neq 0$, we conclude that D^{i-1} may be written as a linear combination of $1, D, \ldots, D^{i-2}$, which contradicts our assumption.

Let $a_0, \dots, a_{p-1} \in K$ be as in the statement, and $E = a_0 + a_1 D + \cdots + a_{p-1} D^{p-1}$. Then $a_0 = E(1) = 0$. Let $a \in K$ be such that $D(a) \neq 0$. By composing both sides with left multiplication by a and by applying Leibniz formula, we obtain

$$aE + E(a) = aa_{p-1} D^{p-1} + ((p-1)a_{p-1} D(a) + aa_{p-2}) D^{p-2} + \cdots$$

Since $1, D, \dots, D^{p-1}$ are linearly independent over K, the coefficients of D^{p-2} on both sides must be equal, so that $(p-1)a_{p-1} D(a) = 0$. We conclude that $a_{p-1} = 0$. Similarly, we prove by induction that $a_{p-2} = \cdots = a_2 = 0$. ■

Proposition A.4 (Hochschild, cf. [2] Lemma 1)**.** *Let A be a ring of characteristic $p > 0$. Then, for $a \in A$, and $D \in \mathrm{Der}(A)$, we have*

$$(aD)^p = a^p D^p + (aD)^{p-1}(a) D.$$

Proof. For any integer $k \geq 2$, one may easily prove by induction that

$$(aD)^k = a^k D^k + \sum_{i=2}^{k-1} a_{k,i} D^i + (aD)^{k-1}(a) D$$

where $a_{k,i} = P_{k,i}(a, D(a), \dots, D^{k-i}(a))$, and $P_{k,i} \in \mathbf{F}_p[T_0, T_1, \dots, T_{k-i}]$ is a polynomial that does not depend on A, a, or D.

Consider the field $K = \mathbf{F}_p(\{T_i\}_{i \in \mathbf{N}})$ and let ∂ be the unique derivation of K satisfying $\partial(T_i) = T_{i+1}$ for every $i \geq 0$. Since $(T_0 \partial)^p$ and $T_0^p \partial^p$ are also derivations of K, by Proposition A.3, we have

$$P_{p,i}(T_0, T_1, \dots, T_{p-i}) = 0$$

for every $2 \leq i \leq p-1$; so that $a_{p,i} = 0$ as well. ■

References

[1] D. Cerveau, A. Lins-Neto, F. Loray, J. Pereira, F. Touzet, *Complex codimension one singular foliations and Godbillon-Vey sequences.* Moscow Mathematical Journal, Volume 7, Number 1, January–March 2007, Pages 21–54.

[2] G. Hochschild, *Simple algebras with purely inseparable splitting fields of exponent 1.* Transactions of the American Mathematical Society, Vol. 79, No. 2 (Jul., 1955), pp. 477-489

[3] N. Jacobson, *Lie Algebras.* Interscience Tracts in Pure and Applied Mathematics, 1962. Reprinted by Dover, 1979.

[4] F. Loray, J. Pereira, F. Touzet, *Singular foliations with trivial canonical class.* Preprint 2015. Available at `https://arxiv.org/abs/1107.1538`.

[5] H. Matsumura, *Commutative Ring Theory.* Cambridge University Press, 1986.

[6] C. Seshadri, *L'opérateur de Cartier. Applications.* Sém. Chevalley 1958-1959, exposé 6, ENS, 1960.