Coefficients of Poincaré series as periods

Tiago J. Fonseca

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Abstract

These are extended notes for a 2 hours talk given at the Galois Theory of Periods seminar, Oxford.

1 Introduction

A general method to construct invariant objects (functions, differential forms, etc.) under the action of some group is to perform some kind of 'averaging' process, where we pick any object and sum over all of its translates under this group. This was Poincaré's main tool for constructing modular functions and modular forms.

Here, *Poincaré series* will be the following functions on the Poincaré half-plane $\mathbb{H} = \{\tau \in \mathbb{C} \mid \Im\tau > 0\}$:

$$P_{m,k,N}(\tau) = \sum_{\gamma \in \Gamma_0(N)_\infty \backslash \Gamma_0(N)} \frac{e^{2\pi i m \gamma\cdot\tau}}{(c\tau + d)^k}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where m is an integer (*index*), $k \geq 4$ is an even natural number (*weight*), and $N \geq 1$ is a natural number (*level*). I recall that

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \mod N \right\}$$

is Hecke's congruence subgroup of level N, and $\Gamma_0(N)_\infty$ is the stabilizer of the cusp ∞, given by the matrices in $\Gamma_0(N)$ such that $c = 0$. When $m \neq 0$, Petersson also constructed 'Poincaré series' in weight $k = 2$ via an analytic continuation trick.

It is easy to prove, at least when $k \geq 4$, that Poincaré series are weakly holomorphic modular forms:

$$P_{m,k,N} \in M_k^!(\Gamma_0(N)),$$

that is, $P_{m,k,N}$ is holomorphic on $\mathbb{H}$, modular of weight k for the group $\Gamma_0(N)$, and 'meromorphic' at all cusps. More precisely, we have the follwoing result.

Theorem (Poincaré, Petersson)**.** *Given k and N, the Poincaré series $P_{m,k,N}$, for $m \neq 0$, generate the vector space $S_k^{!,\infty}(\Gamma_0(N))$ of weakly holomorphic modular forms with vanishing constant Fourier coefficient at every cusp, and holomorphic at every cusp $\neq \infty$. If $m > 0$, then $P_{m,k,N} \in S_k(\Gamma_0(N))$ (space of cusp forms).*

Thus, the theory of modular forms 'reduces' to the study of Poincaré series, but in practice this is not always helpful. For instance, we know that $S_k(\Gamma_0(N))$ is finite-dimensional, therefore $P_{m,k,N}$, for $m > 0$, must satisfy many linear relations. These are still not well understood. Worse, Poincaré series can vanish identically even when $S_k(\Gamma_0(N)) \neq 0$, such as

$$P_{m,4,8} \equiv 0, \qquad m = 2, 4, 6, 8, \ldots,$$

but in general it is hard to know when this happens. For instance, Lehmer's conjecture is equivalent to asserting that $P_{m,12,1} \not\equiv 0$ for every $m > 0$.

From the point of view of Number Theory, we wish to understand the Fourier coefficients $a_n(P_{m,k,N}) \in \mathbb{C}$, defined by

$$P_{m,k,N} = \sum_{n \gg -\infty} a_n(P_{m,k,N})\, q^n, \qquad q = e^{2\pi i \tau}.$$

Questions. What are $a_n(P_{m,k,N})$? Can we give explicit formulas? Are they rational? Algebraic?

Fix $n \geq 1$ and $y > 0$. From

$$a_n(P_{m,k,N}) = \int_0^1 P_{m,k,N}(x+iy) e^{-2\pi i n(x+iy)} dx$$

one can deduce

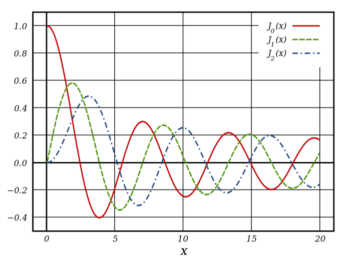
$$a_n(P_{m,k,N}) = \begin{cases} \delta_{m,n} + 2\pi i^k \left(\frac{n}{m}\right)^{\frac{k-1}{2}} \sum_{c\geq 1,\ N|c} \frac{K(m,n;c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right) & m > 0 \\ 2\pi i^k \left(\frac{n}{|m|}\right)^{\frac{k-1}{2}} \sum_{c\geq 1,\ N|c} \frac{K(m,n;c)}{c} I_{k-1}\left(\frac{4\pi\sqrt{|m|n}}{c}\right) & m < 0 \end{cases}$$

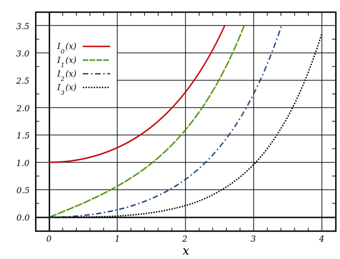
where

$$K(a,b;c) = \sum_{z \in (\mathbb{Z}/c\mathbb{Z})^\times} e^{\frac{2\pi i}{m}(az + bz^{-1})} \in \mathbb{R} \cap \overline{\mathbb{Q}}$$

is a Kloosterman sum, and J and I are Bessel functions.

Remark. Both identities are essentially the same, since $I_r(x) = i^{-r} J_r(ix)$. Note however that these functions J and I Bessel functions have very different behaviour: J oscilates and tends to zero, while I is monotone and grows exponentially.





According to Sarnak, the above series expansion for the Fourier coefficients of a Poincaré series is an 'algebraist's nightmare'. On the other hand, since the growth of Bessel functions are well understood, and since estimating Kloosterman sums is a whole industry in itself, one can derive important analytic consequences from such formulas. For instance, one can improve Hecke's bound:

$$a_n(f) = O_\epsilon(n^{k/2-1/4+\epsilon}), \qquad f \in S_k(\Gamma_0(N)).$$

Negative indexes are also useful. If

$$j = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \cdots \in M_0^!(\Gamma_0(1))$$

is the usual modular invariant, then we have

$$Dj = P_{-1,2,1}, \qquad D = \frac{1}{2\pi i}\frac{d}{d\tau}$$

which yields an assymtoptic formula (Petersson, Rademacher)

$$a_n(j) \sim \frac{e^{4\pi\sqrt{n}}}{\sqrt{2}n^{3/4}}.$$

Unfortunately, this does not clarify the arithmetic nature of Fourier coefficients Poincaré series. The formula $Dj = P_{-1,2,1}$ already shows that such Fourier coefficients can be rational numbers (actually, integers!). There are many other examples, such as $P_{-m,4,8}$ for $m = 2, 4, 6, 8, \ldots$, or

$$P_{-1,4,9} = \frac{1}{q} + 2q^2 - 49q^5 + 48q^8 + 771q^{11} - \cdots.$$

However, these are quite special examples. One might remark, for instance, that the unique normalized newform of weight 4 and level 9 has CM. In general, we expect Fourier coefficients of Poincaré series to be transcendental numbers. For instance, if

$$\Delta = q \prod_{n=1}^{\infty} (1 - q^n)^{24} \in S_{12}(\Gamma_0(1))$$

is the modular discriminant, then

$$P_{1,12,1} = \lambda_\Delta \Delta, \qquad \lambda_\Delta = 2.84028\ldots$$

and λ_Δ seems to be transcendental.

2 Single-valued periods of modular forms

According to Kontsevich and Zagier, whenever we meet a new number, and have convinced ourselves that it is transcendental, we should try to figure out whether it is a *period*. Loosely speaking, a period is an integral of an algebraic differential form. The number λ_Δ above does have an integral representation; this follows from the next result.

Theorem (Petersson)**.** *The pairing*

$$(f, g)_{\mathrm{Pet}} = \int_{\Gamma_0(N)\backslash\mathbb{H}} f(x+iy)\overline{g(x+iy)}y^{k+2}\frac{dxdy}{y^2}$$

defines a Hermitian inner product on $S_{k+2}(\Gamma_0(N))$. For every $m > 0$, we have

$$(f, P_{m,k+2,N})_{\mathrm{Pet}} = \frac{k!}{(4\pi m)^{k+1}} a_n(f).$$

This is actually how Petersson proves the theorem in the last section. In any case, applying it to the identity $P_{1,12,1} = \lambda_\Delta \Delta$, we get

$$\lambda_\Delta (\Delta, \Delta)_{\mathrm{Pet}} = (\Delta, P_{1,12,1})_{\mathrm{Pet}} = \frac{10!}{(4\pi)^{11}}$$

so that

$$\lambda_\Delta^{-1} = \frac{(4\pi)^{11}}{10!} \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} |\Delta(x+iy)|^2 y^{12} \frac{dxdy}{y^2}.$$

Although Δ is a transcendental function, the above integral is indeed a period in the sense of Kontsevich and Zagier; more precisely, it is a *single-valued period*.

Example. Let us explain this in weight 2, where the geometry is simpler. Let

$$f(\tau) = \eta(\tau)^2\eta(11\tau)^2 = q - 2q^2 - q^3 + 2q^4 + q^5 + \cdots \in S_2(\Gamma_0(11))$$

be the unique normalized cusp form in $S_2(\Gamma_0(11))$. Arguing as above, we have

$$P_{1,2,11} = \lambda f, \qquad \text{with} \qquad \lambda^{-1} = 4\pi \int_{\Gamma_0(11)\backslash\mathbb{H}} |f(x+iy)|^2 dxdy.$$

Note that $\Gamma_0(11)\backslash\mathbb{H}$ is the analytification of the open modular curve $Y_0(11)$, which exists as a $\mathbb{Q}$-scheme. If we set

$$\omega_f = f(\tau)\,2\pi i d\tau = f(q) d\log q$$

then the modularity of f implies that ω_f descends to $Y_0(11)^{\text{an}}$. Since it is a cusp form, it extends to a holomorphic 1-form on the compactified modular curve $X_0(11)^{\text{an}}$. Moreover, since the Fourier coefficients of f are rational, the q-expansion principle implies that ω_f is actually an algebraic 1-form defined over $\mathbb{Q}$, i.e., $\omega_f \in \Gamma(X_0(11), \Omega^1_{X_0(11)/\mathbb{Q}})$. We now rewrite

$$\lambda^{-1} = 4\pi \int_{\Gamma_0(11)\backslash\mathbb{H}} |f(x+iy)|^2 dxdy = -\frac{1}{2\pi i}\int_{X_0(11)^{\text{an}}} \omega_f \wedge \overline{\omega_f}$$

Note that $X_0(11)$ is of genus 1. If (γ_1, γ_2) is any basis of $H_1(X_0(11)^{\text{an}}, \mathbb{Z})$ satisfying $\gamma_1 \cdot \gamma_2 = 1$, the 'double copy formula' gives

$$\lambda^{-1} = -\frac{1}{2\pi i}\int_{X_0(N)^{\text{an}}} \omega_f \wedge \overline{\omega_f} = -\frac{1}{2\pi i}(\omega_1\overline{\omega_2} - \omega_2\overline{\omega_1}) = -\frac{\Im(\omega_1\overline{\omega_2})}{\pi}, \qquad \text{where} \qquad \omega_j = \int_{\gamma_j} \omega_f.$$

Thus, we have expressed λ^{-1} in terms of periods of algebraic integrals.

Let us now recall the general formalism of single-valued periods, as in Brown-Dupont. Let H be a 'mixed motive' over a subfield $K \subset \mathbb{R}$. In particular, H has a:

1. Betti realisation H_{B}, a $\mathbb{Q}$-vector space with a real Frobenius involution $F_\infty : H_{\text{B}} \to H_{\text{B}}$ and a weight filtration,
2. de Rham realisation H_{dR}, a K-vector space with a weight and a Hodge filtration

related by a comparison isomorphism

$$\text{comp} : H_{\text{dR}} \otimes_K \mathbb{C} \xrightarrow{\sim} H_{\text{B}} \otimes_{\mathbb{Q}} \mathbb{C}.$$

These are compatible in the sense that they induce a $\mathbb{Q}$-mixed Hodge structure on H_{B}, and the following diagram commutes

$$\begin{array}{ccc} H_{\text{dR}} \otimes_K \mathbb{C} & \xrightarrow{\text{comp}} & H_{\text{B}} \otimes_{\mathbb{Q}} \mathbb{C} \\ {\scriptstyle \text{id}\otimes c_{\text{dR}}}\big\downarrow & & \big\downarrow{\scriptstyle F_\infty \otimes c_{\text{B}}} \\ H_{\text{dR}} \otimes_K \mathbb{C} & \xrightarrow[\text{comp}]{} & H_{\text{B}} \otimes_{\mathbb{Q}} \mathbb{C} \end{array}$$

commutes, where c_{B} and c_{dR} denote the action of complex conjugation on coefficients.

Remark. Basic examples of such mixed motives come from the cohomology $H^n(X)$ of smooth algebraic varieties X over K. Here, $H^n(X)_{\text{dR}} = H^n_{\text{dR}}(X/K)$ is the nth algebraic de Rham cohomology with coefficients in K, and $H^n(X)_{\text{B}} = H^n(X(\mathbb{C}), \mathbb{Q})$ is the nth singular cohomology of the complex manifold $X^{\text{an}} = X(\mathbb{C})$. The comparison isomorphism is given by integration of differential forms, and the involution F_∞ is induced by the continuous map $X(\mathbb{C}) \to X(\mathbb{C})$ given by complex conjugation on $\mathbb{C}$-points.

It follows from the above diagram that $\text{comp}^{-1} \circ (F_\infty \otimes \text{id}) \circ \text{comp}$ is invariant under the complex conjugation $\text{id} \otimes c_{\text{dR}}$, so that there exists an $\mathbb{R}$-linear involution

$$\text{sv} : H_{\text{dR}} \otimes_K \mathbb{R} \longrightarrow H_{\text{dR}} \otimes_K \mathbb{R}$$

whose $\mathbb{C}$-linear extension is $\text{comp}^{-1} \circ (F_\infty \otimes \text{id}) \circ \text{comp}$; this is the *single-valued involution.*

Definition. A *single-valued period* of H is any real number of the form

$$\varphi(\text{sv}(\omega)) \in \mathbb{R}$$

where $\omega \in H_{\text{dR}}$ and $\varphi \in H^\vee_{\text{dR}}$.

Remark. Single-valued periods relate to (Kontsevich-Zagier) periods as follows. Let us fix a K-basis (ω_i) of H_{dR} and a $\mathbb{Q}$-basis (γ_i) of $H_{\mathrm{B}}^{\vee}$. The matrix of the comparion isomorphism comp with respect to these basis is a *period matrix* P. Then, the matrix of sv with respect to (ω_i) is given by

$$S = P^{-1}\overline{P} = P^{-1}F_{\infty}P.$$

The coefficients of S generate the K-linear span of all single-valued periods of H.

We are interested in single-valued periods attached to modular forms. By a theorem of Scholl, there is a pure motive over $\mathbb{Q}$

$$H^1_{\mathrm{cusp}}(\mathcal{Y}_0(N), \mathrm{Sym}^k H^1(\mathcal{E}))$$

of Hodge type $\{(k+1,0),(0,k+1)\}$ whose de Rham realisation is given by the cuspidal de Rham cohomology with coefficients

$$H^1_{\mathrm{dR,cusp}}(\mathcal{Y}_0(N), \mathrm{Sym}^k H^1_{\mathrm{dR}}(\mathcal{E}/\mathcal{Y}_0(N))),$$

where $H^1_{\mathrm{dR}}(\mathcal{E}/\mathcal{Y}_0(N)))$ is endowed with the Gauss-Manin connection. Here, $\mathcal{Y}_0(N)$ denotes the *moduli stack* over $\mathbb{Q}$ attached to the moduli problem with elliptic curves endowed with a cyclic subgroup of order N, and $\mathcal{E}$ denotes the universal elliptic curve over $\mathcal{Y}_0(N)$. Note that the analytification $\mathcal{Y}_0(N)^{\mathrm{an}}$ of $\mathcal{Y}_0(N)$ is isomorphic to the *orbifold* quotient $Y_0(N)\backslash\backslash\mathbb{H}$.

Here is a more concrete description of $H^1_{\mathrm{dR,cusp}}(\mathcal{Y}_0(N), \mathrm{Sym}^k H^1_{\mathrm{dR}}(\mathcal{E}/\mathcal{Y}_0(N)))$.

Theorem (Scholl, Coleman, Brown-Hain, etc.)**.** *We have an exact sequence of* $\mathbb{Q}$*-vector spaces*

$$0 \longrightarrow M^{!,\infty}_{-k}(\Gamma_0(N);\mathbb{Q}) \xrightarrow{D^{k+1}} S^{!,\infty}_{k+2}(\Gamma_0(N);\mathbb{Q}) \longrightarrow H^1_{\mathrm{cusp}}(\mathcal{Y}_0(N), \mathrm{Sym}^k H^1(\mathcal{E}))_{\mathrm{dR}} \longrightarrow 0$$
$$f \mapsto [f(\tau)\omega^k \otimes 2\pi i\, d\tau]$$

where $\omega_\tau = [2\pi i dz]$ *on* $H^1_{\mathrm{dR}}(\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z}))$*. Moreover,* $S_{k+2}(\Gamma_0(N);\mathbb{Q})$ *maps injectively onto the Hodge filtration* $F^{k+1}H^1_{\mathrm{cusp}}(\mathcal{Y}_0(N), \mathrm{Sym}^k H^1(\mathcal{E}))_{\mathrm{dR}}$.

For a subfield $K \subset \mathbb{C}$, we denote by $M^!_k(\Gamma_0(N);K) \subset M^!_k(\Gamma_0(N))$ the K-subspace of weakly holomorphic modular forms whose Fourier coefficients at infinity lie in K, and similarly for $S^{!,\infty}_k(\Gamma_0(N);K)$, etc.

Remark. We could also include Eichler and Shimura in the above list. Actually, a primitive form of the above theorem traces back to Poincaré!

We will see that the Petersson inner product of two cusp forms in $S_{k+2}(\Gamma_0(N);\mathbb{Q})$ is a single-valued period of the motive $H^1_{\mathrm{cusp}}(\mathcal{Y}_0(N), \mathrm{Sym}^k H^1(\mathcal{E}))$. Using Petersson's theorem, it is not hard to conclude that we can always express coefficients of Poincaré series of positive index in terms of single-valued periods of these modular motives. What about Poincaré series of negative index? In general, what are the single-valued periods of $H^1_{\mathrm{cusp}}(\mathcal{Y}_0(N), \mathrm{Sym}^k H^1(\mathcal{E}))$?

3 Coefficients of Poincaré series and single-valued periods

For simplicity, let us denote $H = H^1_{\mathrm{cusp}}(\mathcal{Y}_0(N), \mathrm{Sym}^k H^1(\mathcal{E}))$ and $P_m = P_{m,k+2,N}$.

Theorem. *Let* $\mathbb{Q}(\mathrm{sv})$ *be the field of rationality of* $\mathrm{sv} : H \otimes_{\mathbb{Q}} \mathbb{R} \to H \otimes_{\mathbb{Q}} \mathbb{R}$ *and* $\mathbb{Q}(P)$ *be the field generated by* $a_n(P_m)$ *for* $n > 0$ *and* $m \neq 0$*. Then*

$$\mathbb{Q}(\mathrm{sv}) = \mathbb{Q}(P).$$

We recall that the field of rationality of an $\mathbb{R}$-linear map is the smallest subfield of $\mathbb{R}$ over which it is defined. If S is a matrix of single-valued periods as in last section, then $\mathbb{Q}(\mathrm{sv})$ is the field generated by the coefficients of S.

Thus, Fourier coefficients of Poincaré series and single-valued periods of modular motives are essentially the same thing. A simple consequence of this result is that $\mathbb{Q}(P)$ is finitely generated. This is not obvious, since spaces of weakly holomorphic modular forms are not finite-dimensional. More importantly, writing Fourier coefficients of Poincaré series as periods helps to elucidate their arithmetic nature, as the algebraic relations they satisfy are governed by Grothendieck's period conjecture.

Example. Let us assume that the motive H has rank 2, e.g. $(k+2, N) \in \{(2,11),(4,9),(12,1)\}$, where we can be a bit more precise. The $\mathbb{Q}$-vector space H_{dR} is trivialized by the classes of f and g, where $f \in S_{k+2}(\Gamma_0(N);\mathbb{Q})$ and $g \in S^{!,\infty}_{k+2}(\Gamma_0(N);\mathbb{Q})$. The single-valued period matrix with respect to this basis is of the form

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$$

with $c \neq 0$, $ad - bc = -1$, $d = -a$. Set $\rho = a/c$. Then, for every $m > 0$, we can prove that there exists $h_m \in M^{!,\infty}_{-k}(\Gamma_0(N);\mathbb{Q})$ such that

$$a_n(P_m) = -\frac{k!}{m^{k+1}} a_m(f) a_n(f) c^{-1} \qquad \text{and} \qquad a_n(P_{-m}) = \frac{k!}{m^{k+1}} a_m(f) a_n(f) \rho + r_{m,n}$$

where $r_{m,n} = k! a_m(f) a_n(g)/m^{k+1} + n^{k+1} a_n(h_m) \in \mathbb{Q}$. In particular, P_{-m} has rational Fourier coefficients whenever $a_m(f) = 0$. Now, why $P_{-1,4,9}$ considered above has rational Fourier coefficients? In this case, $f = q - 8q^4 + \cdots \in S_4(\Gamma_0(9))$ has CM by $\mathbb{Q}(\sqrt{-3})$. This implies that the motive H, after extension to $\mathbb{Q}(\sqrt{-3})$ has an extra endomorphism. Its matrix in the basis given by f and g must satisfy

$$\begin{pmatrix} x & y \\ 0 & w \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \overline{x} & \overline{y} \\ 0 & \overline{w} \end{pmatrix}$$

where $x, y, w \in \mathbb{Q}(\sqrt{-3})$. This implies that $\rho = y/(x - \overline{x}) \in \mathbb{Q}(\sqrt{-3}) \cap \mathbb{R} = \mathbb{Q}$.

The proof of the theorem above is based on the theory of harmonic lifts of modular forms and in its relation with the single-valued involution, which yields the following result.

Proposition. *For every $m \neq 0$, we have* $\mathrm{sv}([P_m]) = -[P_{-m}]$ *in* H_{dR}.

Proof. By reinterpreting and (slightly) generalizing results of Bruinier-Funke, Bruinier-Ono-Rhoades, Brown, and Candelori we obtain a commutative diagram

$$\begin{array}{ccc} & H^!_{-k}(\Gamma_0(N)) & \\ \swarrow^{\frac{1}{(4\pi)^{k+1}}\xi_{-k}} & & \searrow^{\frac{1}{k!}D^{k+1}} \\ M^!_{k+2}(\Gamma_0(N)) & & M^!_{k+2}(\Gamma_0(N)) \\ \downarrow & & \downarrow \\ H^1_{\mathrm{dR}}(\mathcal{Y}_0(N), \mathrm{Sym}^k H^1_{\mathrm{dR}}(\mathcal{E}/\mathcal{Y}_0(N))) \otimes_{\mathbb{Q}} \mathbb{C} & \xrightarrow[\mathrm{sv}\otimes c_{\mathrm{dR}}]{} & H^1_{\mathrm{dR}}(\mathcal{Y}_0(N), \mathrm{Sym}^k H^1_{\mathrm{dR}}(\mathcal{E}/\mathcal{Y}_0(N))) \otimes_{\mathbb{Q}} \mathbb{C} \end{array}$$

where $H^!_{-k}(\Gamma_0(N))$ is the space of 'harmonic weak Maass forms of manegeable growth', and $\xi_{-k} = 2i(\Im\tau)^{-k}\overline{\frac{\partial}{\partial\overline{\tau}}}$. Now, a result of Ono-Bringmann shows that, for $m > 0$, there exists $Q_{-m} \in H^!_{-k}(\Gamma_0(N))$ (so-called Maass-Poincaré series) such that $\xi_{-k} Q_{-m} = \frac{(4\pi)^{k+1} m^{k+1}}{k!} P_m$ and $D^{k+1} Q_{-m} = -m^{k+1} P_{-m}$. This, together with the above diagram, proves our statement. □

The proof of our theorem is now essentially linear algebra. Let us sketch its main steps.

Sketch of proof. We start by remarking that there exists a $\mathbb{Q}$-bilinear symplectic pairing $\langle\ ,\ \rangle_{\mathrm{dR}} : H_{\mathrm{dR}} \times H_{\mathrm{dR}} \to \mathbb{Q}$ given by the cup product composed with the trace form. This induces a Hermitian form on $H_{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{C}$ by the formula

$$(\omega, \eta)_{\mathrm{dR}} = -\langle \omega, (\mathrm{sv} \otimes c_{\mathrm{dR}}) \rangle_{\mathrm{dR}}$$

Note that, if $\omega, \eta \in H_{\mathrm{dR}}$, then $(\omega, \eta)_{\mathrm{dR}}$ is a single-valued period. Moreover, over $F^{k+1}H_{\mathrm{dR}} \cong S_{k+2}(\Gamma_0(N))$, one can prove that $(\ ,\ )_{\mathrm{dR}} = (4\pi)^{k+1}(\ ,\ )_{\mathrm{Pet}}$.

Let $m_1, \ldots, m_d > 0$ be such that $(P_{m_1}, \ldots, P_{m_d})$ is a basis of $S_{k+2}(\Gamma_0(N))$. Using the above proposition, we prove that $\mathbb{Q}(P) = \mathbb{Q}(a_n(P_{\pm m_i}) \mid n > 0,\, 1 \leq i \leq d)$. Now, let $f_1, \ldots, f_d, g_1, \ldots, g_d \in S^{!,\infty}_{k+2}(\Gamma_0(N); \mathbb{Q})$ be such that $f_i \in S_{k+2}(\Gamma_0(N))$ and $([f_1], \ldots, [f_d], [g_1], \ldots, [g_d])$ is a symplectic basis of $(H_{\mathrm{dR}}, \langle\ ,\ \rangle_{\mathrm{dR}})$. The single-valued period matrix for this basis can be written in block form:

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$$

and one can prove that $\mathbb{Q}(S) = \mathbb{Q}(A, C)$. Moreover, by definition, we have $C_{ij} = -([f_i], [f_j])_{\mathrm{dR}}$.

Let us write $P_{m_j} = \sum_{i=1}^{d} \Lambda_{ij} f_i$ for some $\Lambda \in \mathrm{GL}_d(\mathbb{R})$. By Petersson's theorem, we have

$$\frac{k!}{m_j^{k+1}} a_{m_j}(f_r) = -\sum_{i=1}^{d} C_{ri} \Lambda_{ij}$$

so that

$$\Lambda = -C^{-1} Q$$

for some matrix $Q \in \mathrm{GL}_d(\mathbb{Q})$. This proves that $a_n(P_{m_i}) \in \mathbb{Q}(C) \subset \mathbb{Q}(\mathrm{sv})$ for every $n > 0$ and every $1 \leq i \leq d$.

By the above proposition, there exists $h_j \in M^{!,\infty}_{-k}(\Gamma_0(N))$ such that

$$P_{-m_j} = -\sum_{i=1}^{d} \Lambda_{ij} \left(\sum_{r=1}^{d} A_{ri} f_r + C_{ri} g_r \right) + D^{k+1} h_j.$$

The principal part at infinity of h_j is easily seen to be contained in $\mathbb{Q}(\mathrm{sv})[q^{-1}]$. Since h_j is holomorphic at every other cusp, and has negative weight, we must have $h \in \mathbb{Q}(\mathrm{sv})(\!(q)\!)$. This proves that $a_n(P_{-m_j}) \in \mathbb{Q}(\mathrm{sv})$ for every $n > 0$ and $1 \leq j \leq d$. We conclude that $\mathbb{Q}(P) \subset \mathbb{Q}(\mathrm{sv})$. The other inclusion is similar. □