Le groupe fondamental d'une courbe elliptique moins un point

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1 Introduction

The purpose of the next talks is to

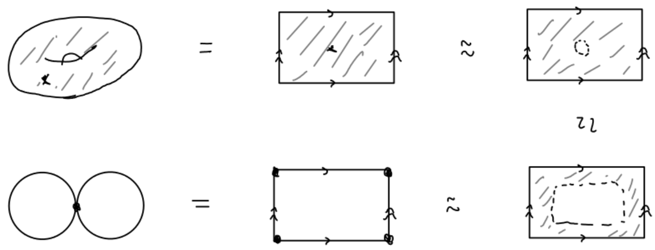
1. give Deligne's base point free definition of $\pi_1^{\mathrm{dR}}(E \setminus \{O\})$, the de Rham fundamental group of a punctured elliptic curve $E \setminus \{O\}$ over a field k of characteristic 0, and

2. show that the Hopf algebra $\mathcal{O}(\pi_1^{\mathrm{dR}}(E \setminus \{O\}))$ is isomorphic to the tensor coalgebra $T^c H^1_{\mathrm{dR}}(E) = \bigoplus_{n \geq 0} H^1_{\mathrm{dR}}(E)^{\otimes n}$ (shuffle product and deconcatenation coproduct).

This is part of a joint project with Nils Matthes aimed at building a motivic theory of elliptic multiple zeta values and elliptic polylogarithms. In particular, we are looking for a good algebraic formalism to deal with families of elliptic curves, so it seems reasonable to avoid the use of base points.

Remark 1. As explained in Nils's talk, the analogous story for $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ is well-known. There is however a subtle difference: unlike the case of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, it will *not* be true that, given two points P and Q in $E \setminus \{O\}$, there is a "canonical de Rham path" from P to Q, i.e., a canonical isomorphism $\pi_1^{\mathrm{dR}}(E \setminus \{O\}; P, Q) \xrightarrow{\sim} \pi_1^{\mathrm{dR}}(E \setminus \{O\})$.

To gain some intuition, let us briefly look at the Betti situation (assume $k \subset \mathbb{C}$). Then $E(\mathbb{C}) \setminus \{O\}$ is a punctured torus, which is homotopically equivalent to the wedge sum of two circles:



In particular, given any $P \in E(\mathbb{C})$, the fundamental group $\Gamma = \pi_1(E(\mathbb{C}) \setminus \{O\}, P)$ is a free group on two generators, say a_1, a_2. Let $[a_1], [a_2]$ denote the corresponding classes in $H_1(E(\mathbb{C}), \mathbb{Z})$.

Recall that the Betti fundamental group $\pi_1^{\mathrm{B}}(E \setminus \{O\}, P)$ is given by the pro-unipotent completion Γ^{un} of Γ, whose corresponding Hopf algebra is given by

$$\mathcal{O}(\Gamma^{\mathrm{un}}) = \operatorname*{colim}_n \operatorname{Hom}(\mathbb{Z}[\Gamma]/J^{n+1}, \mathbb{Q}),$$

where J denotes the augmentation ideal. Now, we can define a $\mathbb{Q}$-linear map

$$H_1(E(\mathbb{C}), \mathbb{Q}) \longrightarrow \lim_n \mathbb{Q}[\Gamma]/J_{\mathbb{Q}}^{n+1}, \qquad [a_i] \longmapsto \log(a_i) = (a_i - 1) - \frac{(a_i - 1)^2}{2} + \cdots.$$

which induces a continuous morphism of complete $\mathbb{Q}$-algebras

$$\hat{T}(H_1(E(\mathbb{C}), \mathbb{Q})) \longrightarrow \lim_n \mathbb{Q}[\Gamma]/J^{n+1},$$

compatible with completed coproducts, antipode, and augmentation. Taking continuous duals, we obtain an isomorphism of Hopf algebras

$$\mathcal{O}(\pi_1^{\mathrm{B}}(E \setminus \{O\}, P)) \xrightarrow{\sim} T^c(H_{\mathrm{B}}^1(E)).$$

Concretely, $T^c(H_{\mathrm{B}}^1(E))$ is the Hopf algebra of non-commutative polynomials $\mathbb{Q}\langle [a_1]^\vee, [a_2]^\vee \rangle$ with the shuffle product and deconcatenation coproduct.

Remark 2. The above isomorphism depends on the choice of generators a_1, a_2 of Γ.

2 In search of a global de Rham fiber functor

In this talk, we focus on our first objective: to give a base point free definition of $\pi_1^{\mathrm{dR}}(E \setminus \{O\})$. Let me stress that we are not looking for some random ad-hoc construction. For instance, we would like this definition to be Tannakian, easily comparable with the usual $\pi_1^{\mathrm{dR}}(E \setminus \{O\}, P)$ defined in terms of the functor 'fiber at P', and ameanable to generalization to the case of families of elliptic curves.

Let us fix a base field k of characteristic 0. Recall from Nils's talk that Deligne defined

$$\pi_1^{\mathrm{dR}}(\mathbb{P}^1 \setminus \{0, 1, \infty\}) = \underline{\mathrm{Aut}}_C^{\otimes}(F)$$

where C is the Tannakian category of unipotent vector bundles with integrable connection over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, and F is the fiber functor

$$F : C \longrightarrow \mathrm{Vect}_k, \qquad (\mathcal{V}, \nabla) \longmapsto H^0(\mathbb{P}^1, \overline{\mathcal{V}}).$$

Here, $(\overline{\mathcal{V}}, \overline{\nabla} : \overline{\mathcal{V}} \longrightarrow \overline{\mathcal{V}} \otimes \Omega^1(\log\{0, 1, \infty\}))$ denotes the canonical extension of $(\mathcal{V}, \nabla)$ to $\mathbb{P}^1$, which exists thanks to unipotency.

Note that F as above is indeed a fiber functor (monoidal, exact, and faithful) because of the following cohomological properties of $\mathbb{P}^1$:

$$H^0(\mathbb{P}^1, \mathcal{O}) = k, \qquad H^1(\mathbb{P}^1, \mathcal{O}) = 0,$$

which imply (see next lemma) that the functor $\mathcal{E} \longmapsto H^0(\mathbb{P}^1, \mathcal{E})$ defines an equivalence between the category of unipotent vector bundles on $\mathbb{P}^1$ and that of finite-dimensional k-vector spaces, a quasi-inverse being given by $E \longmapsto E \otimes \mathcal{O}$.

Lemma 2.1. *Let X be a k-scheme satisfying $H^0(X, \mathcal{O}) = k$ and $H^1(X, \mathcal{O}) = 0$. For every unipotent vector bundle $\mathcal{E}$ on X, the natural $\mathcal{O}$-linear map*

$$H^0(X, \mathcal{E}) \otimes \mathcal{O} \longrightarrow \mathcal{E}$$

is an isomorphism.

Proof. Since $H^0(X, \mathcal{O}) = k$, it is sufficient to prove that $\mathcal{E}$ is constant. This is done by induction on the rank r of $\mathcal{E}$. The base case is trivial: a unipotent line bundle is necessarily isomorphic to $\mathcal{O}$. If $r > 1$, then we can find an exact sequence

$$0 \longrightarrow \mathcal{E}'' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}' \longrightarrow 0$$

where $\mathcal{E}'$ and $\mathcal{E}''$ are unipotent vector bundles of ranks $r', r'' < r$, thus isomorphic to $\mathcal{O}^{\oplus r'}$ and $\mathcal{O}^{\oplus r''}$ by induction hypothesis. Since

$$\mathrm{Ext}^1(\mathcal{O}, \mathcal{O}) = H^1(X, \mathcal{O}) = 0,$$

we obtain

$$\mathrm{Ext}^1(\mathcal{E}', \mathcal{E}'') = \mathrm{Ext}^1(\mathcal{O}^{\oplus r'}, \mathcal{O}^{\oplus r''}) = \mathrm{Ext}^1(\mathcal{O}, \mathcal{O})^{\oplus r' r''} = 0$$

so that

$$\mathcal{E} \cong \mathcal{E}' \oplus \mathcal{E}'' \cong \mathcal{O}^{\oplus r' + r''}.$$

□

A direct analog of this approach certainly will not work in the case of a punctured elliptic curve, since $H^1(E, \mathcal{O})$ does note vanish. One could fix this by finding a k-scheme X satisfying the hypotheses of Lemma 2.1 above, and equipped with a faithfully flat morphism $\pi : X \longrightarrow E$. If such a diagram

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle \pi} \\ E \setminus \{O\} & \hookrightarrow & E \end{array} \tag{D}$$

exists, then

$$F : (\mathcal{V}, \nabla) \longmapsto H^0(X, \pi^* \overline{\mathcal{V}})$$

defines a fiber functor (see remark below) on the category C of unipotent vector bundles with integrable connection on $E \setminus \{O\}$, and we can define

$$\pi^1_{\mathrm{dR}}(E \setminus \{O\}) = \underline{\mathrm{Aut}}^{\otimes}_C(F).$$

Remark 3. If $f : X \longrightarrow Y$ is a faithfully flat morphism of schemes, then the pullback f^* defines a faithful exact functor from the category of quasicoherent sheaves on Y to that of X.

Such putative fundamental group is easily relatable to the usual $\pi_1^{\mathrm{dR}}(E \setminus \{O\}, P)$ for a given point P of $E \setminus \{O\}$: any lift x of P to X gives rise to an isomorphism

$$\pi^1_{\mathrm{dR}}(E \setminus \{O\}) \times_k k_x \xrightarrow{\sim} \pi_1(E \setminus \{O\}, P) \times_{k_P} k_x$$

induced by the the specialization

$$H^0(X, \pi^* \overline{\mathcal{V}}) \longrightarrow (\pi^* \overline{\mathcal{V}})(x) \cong \overline{\mathcal{V}}(P) \otimes_{k_P} k_x.$$

Remark 4. We would like moreover $\pi : X \longrightarrow E$ to behave well in families, but we will not discuss these matters in this talk.

3 The universal vectorial extension of an abelian variety

We now explain how a classical construction in the theory of algebraic groups allows us to find a diagram (D). This idea is due to Deligne.

Let k be a field. We work more generally with abelian varieties A over k (projective, geometrically connected group scheme) since the theory is the same, but there is no harm in thinking only in the case of an elliptic curve

Definition 3.1. A *vector group* over k is a k-group scheme associated to a vector space V over k; by abuse, we also denote it by V, but it's formally given by $\mathrm{Spec}(\mathrm{Sym}\, V^\vee)$.

Definition 3.2. A *vectorial extension* of an abelian variety A over k is an extension of the form

$$0 \longrightarrow V \longrightarrow E \longrightarrow A \longrightarrow 0 \tag{E}$$

in the category of commutative k-group schemes.

The exactness of (E) means that $E \longrightarrow A$ is a faithfully flat morphism of commutative k-group schemes and $V \longrightarrow E$ induces an isomorphism onto its kernel, the fiber over the identity of A.

Now, given a morphism of vector groups $V \longrightarrow V'$ and an extension E of A be V, we obtain an extension E' of A by V' by *pushout*:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & V' & \longrightarrow & E' & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

so that $E' = E \sqcup_V V'$ in the category of commutative k-group schemes.

Definition 3.3. A vectorial extension (E) is *universal* if any other vectorial extension of A

$$0 \longrightarrow V' \longrightarrow E' \longrightarrow A \longrightarrow 0$$

is obtained by pushing out (E) along a unique morphism $V \longrightarrow V'$.

In other words, the map

$$\mathrm{Hom}_k(V, V') \longrightarrow \mathrm{Ext}(A, V')$$

given by the pushout is bijective. The existence of a universal vectorial extension is guaranteed by the following fundamental result.

Proposition 3.4 (Barsotti, Rosenlicht, Serre)**.** *For every abelian variety A over k, there is a canonical isomorphism of abelian groups*

$$\mathrm{Ext}(A, \mathbb{G}_a) \xrightarrow{\sim} H^1(A, \mathcal{O}),$$

where Ext *is taken in the category of commutative k-group schemes.*

The map associates to an extension the class of the $\mathbb{G}_a$-torsor over A it defines. The statement is non-trivial; for instance, one has to prove that any $\mathbb{G}_a$-torsor over A admits the structure of a k-group scheme.

It follows from the above proposition that, for any k-vector group V,

$$\mathrm{Ext}(A, V) \cong H^1(A, \mathcal{O} \otimes_k V) \cong H^1(A, \mathcal{O}) \otimes_k V \cong \mathrm{Hom}_k(H^1(A, \mathcal{O})^\vee, V).$$

The universal vectorial extension of A is then the extension

$$0 \longrightarrow H^1(A,\mathcal{O})^\vee \longrightarrow A^\natural \longrightarrow A \longrightarrow 0$$

corresponding to the identity map of $H^1(A,\mathcal{O}_A)^\vee$ under the above identifications. ($A^\natural$ is pronounced 'A natural', I think.)

Universal vectorial extensions can also be thought as moduli spaces for line bundles with connections. This leads to the fact that their Lie algebra is the de Rham homology of the given abelian variety.

Theorem 3.5 (Grothendieck, Mazur-Messing)**.** *Let A be an abelian variety over a field k. The universal vectorial extension of $A^\vee = \mathrm{Pic}^0(A)$ can be identified with*

$$0 \longrightarrow H^0(A,\Omega^1) \longrightarrow \mathrm{Pic}^\natural(A) \longrightarrow A^\vee \longrightarrow 0$$

where $\mathrm{Pic}^\natural(A)$ is the moduli space of line bundles algebraically equivalent to 0 endowed with an integrable connection (L,∇) on A.[1] *The map on the left associates ω to the class of $(\mathcal{O}, d+\omega)$, and the map on the left forgets the connection.*

The proof is quite involved. The idea is first to interpret $\mathrm{Pic}^\natural(A)$ as the group of 'rigidified extensions' of A by $\mathbb{G}_m$, then to prove that this coincides with the universal vectorial extension by reducing to positive characteristic and using previous results on the universal vectorial extension of p-divisible groups.

Using the above theorem, and the natural biduality $A \cong A^{\vee\vee}$, one can show that $A^\natural(k)$ can be identified with the first hypercohomology of the multiplicative de Rham complex on $A^\vee$

$$0 \longrightarrow \mathcal{O}^\times \xrightarrow{d\log} \Omega^1 \longrightarrow \Omega^2 \longrightarrow \cdots$$

An infinitesimal version of this fact yields the next proposition.

Proposition 3.6. *Let A be an abelian variety over a field k. Then we can identify*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Lie}\, H^1(A,\mathcal{O})^\vee & \longrightarrow & \mathrm{Lie}\, A^\natural & \longrightarrow & \mathrm{Lie}\, A & \longrightarrow & 0 \\
 & & \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \\
0 & \longrightarrow & H^1(A,\mathcal{O})^\vee & \longrightarrow & H^1_{\mathrm{dR}}(A)^\vee & \longrightarrow & H^0(A,\Omega^1)^\vee & \longrightarrow & 0
\end{array}$$

where the bottom row is the dual of the Hodge exact sequence of A.

Remark 5. If $k=\mathbb{C}$, then we have the following short exact sequence of short exact sequences of commutative complex analytic Lie groups

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & H^1(A,\mathcal{O})^\vee & \longrightarrow & H^1(A,\mathcal{O})^\vee & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H_1(A(\mathbb{C}),\mathbb{Z}) & \xrightarrow{\mathrm{comp}^\vee} & H^1_{\mathrm{dR}}(A)^\vee & \xrightarrow{\exp} & A^\natural(\mathbb{C}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H_1(A(\mathbb{C}),\mathbb{Z}) & \longrightarrow & H^0(A,\Omega^1)^\vee & \longrightarrow & A(\mathbb{C}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

[1] In characteristic 0, a line bundle L over A with an integrable connection ∇ is automatically algebraically equivalent to 0.

In particular, if E is an elliptic curve and $\Lambda \subset \mathbb{C}$ is a lattice such that $E(\mathbb{C}) \cong \mathbb{C}/\Lambda$, then $E^\natural(\mathbb{C}) \cong \mathbb{C}^2/\Lambda^\natural$, where

$$\Lambda^\natural = \{(\omega, \eta) \in \mathbb{C}^2 \mid \omega \in \Lambda,\, \eta = \int_0^\omega \wp_\Lambda(z)dz\} \subset \mathbb{C}^2.$$

However, one must be very careful in using analytic arguments on $A^\natural$, as it is *not* projective, and GAGA fails badly.

We now move to the key cohomological properties of the universal vectorial extension.

Theorem 3.7. *Let A be an abelian variety over a field k of* charactetistic 0. *Then*

$$H^0(A^\natural, \mathcal{O}) = k \qquad \textit{and} \qquad H^1(A^\natural, \mathcal{O}) = 0.$$

A k-group scheme G satisfying $H^0(G, \mathcal{O}) = k$ is called *anti-affine*. The proof that $A^\natural$ is anti-affine below is due to Brion.

Proof. Let $B = H^0(A^\natural, \mathcal{O})$ and $G = \operatorname{Spec} B$. Note that G is a commutative affine k-group scheme and that the canonical map

$$\varphi : A^\natural \longrightarrow G$$

is an epimorphism of commutative k-group schemes. Let $V_A = H^1(A, \mathcal{O})^\vee$ (as a vector group), and $V = \varphi(V_A)$, so that φ restricts to an epimorphism $\varphi|_{V_A} : V_A \longrightarrow V$. Since V_A is a vector group, and k is of characteristic 0, $\ker(\varphi|_{V_A})$ is a vector subgroup of V_A; thus V is also a vector group. Note that φ induces an epimorphism $A \longrightarrow G/V$. Since A is projective and G/V is affine, we must have $G = V$, so that we obtain the following commutative diagram

$$\begin{array}{ccc} A^\natural & \longrightarrow & V \\ \uparrow & \nearrow & \\ V_A & & \end{array}$$

Let

$$0 \longrightarrow V \longrightarrow E \longrightarrow A \longrightarrow 0$$

be the unique extension of A by V corresponding to $V_A \longrightarrow V$. Since it is a pushout, the map $A^\natural \longrightarrow V$ induces a retraction $E \longrightarrow V$, so that it splits. This means that the epimorphism $V_A \longrightarrow V$ is the zero map. Thus $B = k$.

The short exact sequence of commutative k-group schemes

$$0 \longrightarrow V_A \longrightarrow A^\natural \longrightarrow A \longrightarrow 0$$

induces an exact sequence of abelian groups

$$\begin{gathered} 0 \longrightarrow \operatorname{Hom}(A, \mathbb{G}_a) \longrightarrow \operatorname{Hom}(A^\natural, \mathbb{G}_a) \longrightarrow \operatorname{Hom}(V_A, \mathbb{G}_a) \xrightarrow{\delta} \\ \operatorname{Ext}(A, \mathbb{G}_a) \longrightarrow \operatorname{Ext}(A^\natural, \mathbb{G}_a) \longrightarrow \operatorname{Ext}(V_A, \mathbb{G}_a) \end{gathered}$$

with δ being the same map given by the universal property of $A^\natural$. Since k is of characteristic 0, we have $\operatorname{Ext}(\mathbb{G}_a, \mathbb{G}_a) = 0$, so that $\operatorname{Ext}(V_A, \mathbb{G}_a) = 0$. Since δ is an isomorphism, we conclude that $\operatorname{Ext}(A^\natural, \mathbb{G}_a) = 0$. This implies that $H^1(A^\natural, \mathcal{O}) = 0$ (I think that, for every commutative anti-affine G, the natural map $\operatorname{Ext}(G, \mathbb{G}_a) \longrightarrow H^1(G, \mathcal{O})$ is surjective). □

We can now give a good definition of $\pi_1^{\mathrm{dR}}(E \setminus \{O\})$ for an elliptic curve E over a field of characteristic 0. Let $\pi : E^\natural \longrightarrow E$ be the universal vectorial extension of E.

Definition 3.8 (Deligne)**.** We define

$$\pi_1^{\mathrm{dR}}(E \setminus \{O\}) = \underline{\mathrm{Aut}}_C^{\otimes}(F)$$

where C is the category of unipotent vector bundles with connection on $E \setminus \{O\}$ and F is the fiber functor

$$(\mathcal{V}, \nabla) \longmapsto H^0(E^\natural, \pi^* \overline{\mathcal{V}}).$$

Let me finish with the following annoying question. The universal vectorial extension $E^\natural$ of an elliptic curve E over k is an algebraic group of dimension 2; in particular, it is quasi-projective. Given a Weierstrass equation $y^2 = 4x^3 - ux - v$ for E, can we deduce an explicit equation for $E^\natural$?