Le groupe fondamental d'une courbe elliptique moins un point, II

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1 Introduction

Let (E, O) be an elliptic curve over a field k of characteristic 0. Last time we defined

$$\pi_1^{\mathrm{dR}}(E \setminus \{O\}) = \underline{\mathrm{Aut}}_C^{\otimes}(F)$$

where C is the category of unipotent vector bundles with integrable connection and F is the fiber functor

$$(\mathcal{V}, \nabla) \longmapsto H^0(E^\natural, \pi^*\overline{\mathcal{V}}).$$

Here, $\overline{\mathcal{V}}$ denotes the canonical extension of $\mathcal{V}$ to E and

$$0 \longrightarrow H^1(E, \mathcal{O})^\vee \longrightarrow E^\natural \xrightarrow{\pi} E \longrightarrow 0 \tag{UVE}$$

the *universal vectorial extension* of E. Today, we will see how to compute this fundamental group:

Theorem 1.1. *There is an isomorphism of Hopf algebras over k*

$$\mathcal{O}(\pi_1^{\mathrm{dR}}(E \setminus \{O\})) \cong T^c(H^1_{\mathrm{dR}}(E)),$$

where

$$T^c(H^1_{\mathrm{dR}}(E)) = \bigoplus_{n \geq 0} H^1_{\mathrm{dR}}(E)^{\otimes n}$$

is equipped with the shuffle product and deconcatenation coproduct.

Let D denote the vector group $\pi^{-1}(O) \cong H^1(E, \mathcal{O})^\vee \cong \mathbb{G}_a$ seen as a divisor in $E^\natural$. Given a unipotent $(\mathcal{V}, \nabla)$ over $E \setminus \{O\}$, the pullback $\pi^*(\overline{\mathcal{V}}, \overline{\nabla})$ of is a vector bundle over $E^\natural$ with integrable connection with logarithmic singularities along D:

$$\pi^*\overline{\nabla} : \pi^*\overline{\mathcal{V}} \longrightarrow \pi^*\overline{\mathcal{V}} \otimes \Omega^1(\log D).$$

In order to understand better how this connection looks like, we are naturally lead to the study of logarithmic differential forms $E^\natural$.

2 Logarithmic differential forms on the universal vectorial extension

Recall that $E^\natural$ is a 2-dimensional smooth k-group scheme. By extending a cotangent vector at the identity to a global invariant differential form we obtain an isomorphism

$$(\operatorname{Lie} E^\natural)^\vee \otimes \mathcal{O} \cong \Omega^1.$$

Since $H^0(E^\natural, \mathcal{O}) = k$, we get

$$H^0(E^\natural, \Omega^1) \cong (\operatorname{Lie} E^\natural)^\vee \cong H^1_{\mathrm{dR}}(E).$$

Similarly, we obtain

$$H^0(E^\natural, \Omega^2) \cong \bigwedge^2 H^0(E^\natural, \Omega^1) \cong H^2_{\mathrm{dR}}(E).$$

Let $i : D \longrightarrow E^\natural$ denote the inclusion. The dual of the exact sequence Lie (UVE) is

$$0 \longrightarrow H^0(E, \Omega^1) \xrightarrow{\pi^*} H^0(E^\natural, \Omega^1) \xrightarrow{i^*} H^0(D, \Omega^1) \longrightarrow 0$$

which corresponds to the Hodge exact sequence on $H^1_{\mathrm{dR}}(E)$ under the natural identifications.

Every $\alpha \in H^0(E^\natural, \Omega^1)$ such that $i^*\alpha \neq 0$ determines a coordinate t on D, so that $D = \operatorname{Spec} k[t]$. Indeed, we have we have

$$\mathcal{O}(D) \cong \operatorname{Sym} H^1(E, \mathcal{O})$$

so it suffices to take $t = i^*\alpha$ under the identification $H^0(D, \Omega^1) \cong H^1(E, \mathcal{O})$.

Theorem 2.1. *Let $\{\nu, \omega_0\}$ be a k-basis of $H^0(E^\natural, \Omega^1)$ such that $i^*\omega_0 = 0$, and denote by t the coordinate on D determined by ν. Then there exists a unique family $\{\omega_n\}_{n\geq 1} \subset H^0(E^\natural, \Omega^1(\log D))$ such that, for every $n \geq 1$,*

1. $\operatorname{Res}_D(\omega_n) = t^n/n!$,
2. $d\omega_n = \omega_{n-1} \wedge \nu$,
3. $\omega_n \wedge \omega_0 = 0$.

Moreover, $\{\nu, \omega_n\}_{n\geq 0}$ is a k-basis of $H^0(E^\natural, \Omega^1(\log D))$, and $\{\nu\wedge\omega_n\}_{n\geq 0}$ is a k-basis of $H^0(E^\natural, \Omega^2(\log D))$.

The proof of this theorem goes by a careful inspection of the residue exact sequence:

$$0 \longrightarrow H^0(E^\natural, \Omega^1) \longrightarrow H^0(E^\natural, \Omega^1(\log D)) \xrightarrow{\operatorname{Res}_D} H^0(D, \mathcal{O}) \longrightarrow 0.$$

Note that Res_D is indeed surjective because Ω^1 is free and $H^1(E^\natural, \mathcal{O}) = 0$.

Corollary 2.2. *Every global 2-form on $E^\natural$ is exact.*

Instead of describing it in detail, it is perhaps more useful to understand what is happening in the analytic picture. Take $k \subset \mathbb{C}$ and choose an identification $E(\mathbb{C}) \cong \mathbb{C}/\Lambda$. Then recall that $E^\natural(\mathbb{C}) \cong \mathbb{C}^2/\Lambda^\natural$ where

$$\Lambda^\natural = \{(\omega, \eta) \in \mathbb{C}^2 \mid \omega \in \Lambda, \eta = \int_0^\omega \wp(z)dz\}.$$

Under these identifications, $\pi : E^\natural(\mathbb{C}) \longrightarrow E(\mathbb{C})$ is simply the projection on the first coordinate (note that $\Lambda^\natural \cong \Lambda$ as abelian groups).

One can show that, over the punctured elliptic curve $E \setminus \{O\}$, the sequence (UVE) splits (algebraically), so we can write

$$E^\natural \setminus D \cong (E \setminus \{O\}) \times D.$$

Under this splitting, one can show that $E^\natural(\mathbb{C}) \setminus D(\mathbb{C})$ is uniformised by

$$(z, w) \longmapsto (\wp(z), \wp'(z), \zeta(z) - w)$$

Given all these identifications, we can take $\omega_0 = dz$, $\nu = dw$, and $\omega_n = f_n(z, w)dz$ given by the coefficients of the formal series (Kronecker function)

$$e^{-wT} \frac{\sigma(z+T)}{\sigma(z)\sigma(T)} = \sum_{n \geq 0} f_n(z, w) T^n.$$

3 Computing the fundamental group

For simplicity, let us denote

$$A^\bullet = H^0(E^\natural, \Omega^\bullet(\log D))$$

so that we have a differential graded algebra

$$0 \longrightarrow A^0 \longrightarrow A^1 \longrightarrow A^2 \longrightarrow 0$$

with $A^0 = k$. Note that A^1 and A^2 are infinite-dimensional!

Recall from last talk that the cohomological properties $H^0(E^\natural, \mathcal{O}) = k$ and $H^1(E^\natural, \mathcal{O}) = 0$ imply that every unipotent vector bundle $\mathcal{V}$ on $E^\natural$ is trivial, canonically isomorphic to $H^0(E^\natural, \mathcal{V}) \otimes \mathcal{O}$. Therefore, in what follows, we restrict our attention to vector bundles of the form

$$V \otimes \mathcal{O}$$

with V a finite dimensional k-vector space.

A logarithmic connection on $V \otimes \mathcal{O}$ is determined by a unique k-linear map

$$\nabla : V \longrightarrow V \otimes A^1$$

which also corresponds to some

$$\omega \in \mathrm{End}(V) \otimes A^1$$

(because V is finite dimensional) via

$$\nabla = d + \omega.$$

Lemma 3.1. *The connection is integrable if and only if* ω *satisfies*

$$d\omega + \omega \wedge \omega = d\omega + \frac{1}{2}[\omega, \omega] = 0$$

in $\mathrm{End}(V) \otimes A^2$*, and* $(V \otimes \mathcal{O}, \nabla)$ *is unipotent if and only if there exists a filtration*

$$0 = V_0 \subset V_1 \subset \cdots \subset V_m = V$$

such that

$$\nabla V_i \subset V_{i-1} \otimes A^1$$

for every $1 \leq i \leq m$*. In this case, we say that the* index of unipotency *is* $\leq m$*.*

Suppose that $(V \otimes \mathcal{O}, \nabla)$ is unipotent. Then we obtain a k-linear map

$$\rho : V \longrightarrow V \otimes T^c(A^1), \qquad v \longmapsto \sum_{n \geq 0} \nabla^{(n)} v$$

where

$$T^c(A^1) = \bigoplus_{n \geq 0} (A^1)^{\otimes n}$$

and $\nabla^{(n)}$ denotes the composition

$$V \xrightarrow{\nabla} V \otimes A^1 \xrightarrow{\nabla \otimes \mathrm{id}} V \otimes (A^1)^{\otimes 2} \longrightarrow \cdots \xrightarrow{\nabla \otimes \mathrm{id}} V \otimes (A^1)^{\otimes n}$$

Note that ρ is indeed well defined because $\nabla^{(n)} = 0$ for $n \geq m$ if $(V \otimes \mathcal{O}, \nabla)$ has index of unipotency $\leq m$. We denote

$$\omega^{(n)} \in \mathrm{End}(V) \otimes (A^1)^{\otimes n}$$

the differential form corresponding to $\nabla^{(n)}$.

Lemma 3.2. *Let us endow $T^c(A^1)$ with the coalgebra structure given by the deconcatenation coproduct:*

$$T^c(A^1) \longrightarrow T^c(A^1) \otimes T^c(A^1), \qquad [\alpha_1 | \cdots | \alpha_n] \longmapsto \sum_{i=0}^{n} [\alpha_1 | \cdots | \alpha_i] \otimes [\alpha_{i+1} | \cdots | \alpha_n].$$

Then, ρ defined above gives V the structure of a $T^c(A^1)$-comodule. Moreover,

$$(V \otimes \mathcal{O}, \nabla) \longmapsto (V, \rho)$$

is an equivalence between the category of unipotent $(V \otimes \mathcal{O}, \nabla)$ (not necessarily integrable) and that of $T^c(A^1)$-comodules.

Sketch of proof. To check that (V, ρ) is a $T^c(A^1)$-comodule, it suffices to check that the corresponding k-linear map

$$\mathrm{End}(V)^\vee \longrightarrow T^c(A^1), \qquad \psi \longmapsto \sum_{n \geq 0} \langle \psi, \omega^{(n)} \rangle$$

is a morphism of coalgebras, which follows from the fact that

$$\langle \psi, \omega^{(n)} \rangle = \langle \ , \omega \rangle^{\otimes n} (\Delta^{(n)} \psi)$$

where Δ denotes the coproduct on $\mathrm{End}(V)^\vee$, and $\Delta^{(n)} : \mathrm{End}(V)^\vee \longrightarrow (\mathrm{End}(V)^\vee)^{\otimes n}$ the nth iterated coproduct.

That $(V \otimes \mathcal{O}) \longmapsto (V, \rho)$ is an equivalence of categories follows from the universal property of $T^c(A^1)$ as the conilpotent cofree coalgebra (see remark below). $\square$

Remark 1. Here's the general picture. By the same formula, *every* connection $\nabla : V \longrightarrow V \otimes A^1$ gives a $C(A^1)$-comodule structure ρ on V, where $C(A^1)$ denotes the cofree coalgebra. One then shows that $T^c(A^1) \subset C(A^1)$ is the conilpotent[1] cofree coalgebra and that, for a unipotent connection, T factors through a $T^c(A^1)$ comodule structure.

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[1] A reduced coalgebra $C = k \oplus \overline{C}$ is *conilpotent* if its coradical filtration $F^n C = k \oplus \{x \in \overline{C} \mid \overline{\Delta}^{(n)}(x) = 0\}$ is exhaustive. Here, $\overline{\Delta} : x \longmapsto \Delta(x) - 1 \otimes x - x \otimes 1$ denotes the reduced coproduct.

Let us now consider the *reduced bar complex*:

$$B^\bullet = T^c(\sigma_{\geq 1}A^\bullet[1]) = T^c(A^1 \longrightarrow A^2),$$

so that

$$B^0 = T^c(A^1)$$

and the derivation in degree 0 is given by

$$D : B^0 \longrightarrow B^1, \qquad [\alpha_1|\cdots|\alpha_n] \longmapsto -\sum_{j=1}^{n}[\alpha_1|\cdots|d\alpha_j|\cdots|\alpha_n] - \sum_{j=1}^{n-1}[\alpha_1|\cdots|\alpha_j \wedge \alpha_{j+1}|\cdots|\alpha_n].$$

Since D is a derivation for the shuffle product, which is compatible with the deconcatenation coproduct, $H^0(B^\bullet) = \ker(D)$ is a subcoalgebra of $T^c(A^1)$. Better: it's a sub Hopf algebra.

Lemma 3.3. *Let $(V \otimes \mathcal{O}, \nabla)$ be unipotent. Then $(V \otimes \mathcal{O}, \nabla)$ is integrable if and only if $\rho : V \longrightarrow V \otimes T^c(A^1)$ factors through $V \otimes H^0(B^\bullet)$.*

Sketch of proof. This is a computation. We first remark that it suffices to show that

$$\omega_\rho = \sum_{n\geq 0} \omega^{(n)} \in \mathrm{End}(V) \otimes H^0(B^\bullet).$$

We then check directly from the integrability condition that

$$(\mathrm{id} \otimes D)(\omega_\rho) = 0.$$

□

Thus,

$$(V \otimes \mathcal{O}, \nabla) \longmapsto (V, \rho)$$

induces an equivalence between the categories of unipotent vector bundles with integrable connection on $E^\natural \setminus D$ with the category of $H^0(B^\bullet)$-comodules. Finally, we use the computations of the last section to prove:

Theorem 3.4. *The Hopf algebra $H^0(B^\bullet)$ is equal to $T^c(H^0(\sigma_{\geq 1}A^\bullet[1])) \cong T^c(H^0(E^\natural, \Omega^1))$.*

Proof. Since

$$H^0(\sigma_{\geq 1}A^\bullet[1]) = \ker(d : A^1 \longrightarrow A^2) \subset A^1$$

we have

$$T^c(H^0(\sigma_{\geq 1}A^\bullet[1])) \subset T^c(A^1)$$

□