Elliptic KZB equations via the universal vector extension

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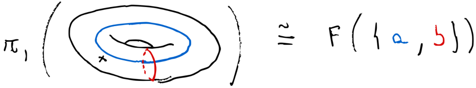
June 2022 - INI, Cambridge

- Joint work in progress with **Nils Matthes**.
- Study unipotent fundamental groups of punctured elliptic curves, e.g. $E \setminus \{O\}$, $E \setminus E[N]$, ...



- Make a case for the universal vector extension of an elliptic curve as the right framework to study the unipotent de Rham fundamental group (Deligne, Enriquez–Etingof).
- Applications to algebraicity and rationality problems for the universal KZB equations.

Unipotent De Rham fundamental group

- X/k smooth variety over a field of characteristic zero.
- $U(X)$ category of unipotent flat vector bundles:

$$(\mathcal{V}, \nabla), \qquad \nabla : \mathcal{V} \to \Omega^1_{X/k} \otimes \mathcal{V}, \qquad \nabla^2 = 0$$

  with

$$0 = (\mathcal{V}_0, \nabla_0) \subset (\mathcal{V}_1, \nabla_1) \subset \cdots \subset (\mathcal{V}_n, \nabla_n) = (\mathcal{V}, \nabla)$$

  such that

$$(\mathcal{V}_i, \nabla_i)/(\mathcal{V}_{i-1}, \nabla_{i-1}) \cong (\mathcal{O}_X \otimes W_i, d_{X/k} \otimes id)$$

- $U(X)$ is Tannakian and every $x \in X(k)$ defines a fibre functor

$$\omega_x : U(X) \to Vect_k, \qquad (\mathcal{V}, \nabla) \mapsto x^*\mathcal{V}$$

- We set

$$\pi_1^{dR}(X,x) = \underline{Aut}^{\otimes}_{U(X)}(\omega_x)$$

  Affine group scheme over k, pro-unipotent.
- Variant with two base points $b, x \in X(k)$:

$$\pi_1^{dR}(X,b,x) = \underline{Isom}^{\otimes}_{U(X)}(\omega_b,\omega_x)$$

Example ($X = \mathbb{P}^1_k \setminus \{0,1,\infty\}$)

$$\mathcal{O}(\pi_1^{dR}(X,x)) \cong T^c H^1_{dR}(X) = \bigoplus_{n\geq 0} H^1_{dR}(X)^{\otimes n}$$

where

$$H^1_{dR}(X) = H^0(\mathbb{P}^1_k, \Omega^1(\log\{0,1,\infty\})) = k\frac{dz}{z} \oplus k\frac{dz}{1-z}.$$

Nice properties: independence of base points, simple poles.

- Every unipotent flat vector bundle on $X = \mathbb{P}^1_k \setminus \{0, 1, \infty\}$ is canonically isomorphic to some

$$(\mathcal{O}_X \otimes V, d - \frac{dz}{z} \otimes A_0 - \frac{dz}{1-z} \otimes A_1)$$

with $A_0, A_1 \in End_k(V)$ "simultaneously nilpotent".

- Proof: consider the canonical extension

$$\overline{\nabla} : \overline{\mathcal{V}} \to \Omega^1_{\mathbb{P}^1_k}(\log\{0, 1, \infty\}) \otimes \overline{\mathcal{V}}$$

and apply

$$\left.\begin{aligned} H^0(\mathbb{P}^1_k, \mathcal{O}) &= k \\ H^1(\mathbb{P}^1_k, \mathcal{O}) &= Ext^1(\mathcal{O}, \mathcal{O}) = 0 \end{aligned}\right\} \text{Deligne's good conditions}$$

Cette dernière hypothèse, très restrictive, est vérifiée si X est rationnelle.

- No hope for an elliptic curve: $H^1(E, \mathcal{O}) \neq 0$.
- Let $\mathcal{V}$ be the vector bundle on $E^{an} = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ given by

$$(m + n\tau) \cdot (v_1, v_2, z) = (v_1 + nv_2, v_2, z + m + n\tau)$$

$$\begin{array}{ccc} \mathbb{C}^2 \times \mathbb{C} & & \mathcal{V} \\ \downarrow & \overset{\text{mod } \mathbb{Z}+\tau\mathbb{Z}}{\rightsquigarrow} & \downarrow \\ \mathbb{C} & & E^{an} \end{array}$$

- A splitting

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{V} \overset{\curvearrowleft}{\longrightarrow} \mathcal{O} \longrightarrow 0$$

corresponds to a function $r : E^{an} \to \mathbb{C}$ satisfying

$$r(z + m + n\tau) = r(z) + n.$$

- No such holomorphic r.

- Real-analytic: $r(z) = Im(z)/Im(\tau)$ (cf. Brown–Levin)

> 3.5. **Massey products on $\mathcal{E}^{(n)}$.** We use the Eisenstein-Kronecker series F to write down some explicit one-forms on $\mathcal{E}^{(n)}$. First consider a single elliptic curve $\mathcal{E}^{\times}$ with coordinate ξ as above. Write $\xi = s + r\tau$, where $r, s \in \mathbb{R}$ and τ is fixed, and let $\omega = d\xi$ and $\nu = 2\pi i dr$. The classes $[\omega], [\nu]$ form a basis for $H^1(\mathcal{E}^{\times}; \mathbb{C})$.
>
> **Lemma 6.** *The form $\Omega(\xi;\alpha) = \mathbf{e}(\alpha r)F(\xi;\alpha)d\xi$ is invariant under elliptic transformations $\xi \mapsto \xi + \tau$ and $\xi \mapsto \xi + 1$, and satisfies $d\Omega(\xi;\alpha) = \nu\alpha \wedge \Omega(\xi;\alpha)$.*

- How to algebraize? Consider $\mathbb{C}^2$ with coordinates (z, r), and lift the action of $\mathbb{Z} + \mathbb{Z}\tau$ by

$$(m + n\tau) \cdot (z, r) = (z + m + n\tau, r + n).$$

- The quotient $\mathbb{C}^2/(\mathbb{Z} + \mathbb{Z}\tau)$ has a natural structure of algebraic variety such that the projection on the first coordinate

$$\mathbb{C}^2/(\mathbb{Z} + \mathbb{Z}\tau) \to \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$$

is algebraic!

- In fact,

$$\mathbb{C}^2/(\mathbb{Z}+\mathbb{Z}\tau) \cong H^1_{dR}(E^{an})/H^1(E^{an},\mathbb{Z}) \cong H^1(E^{an},\mathbb{C}^\times)$$

  classifies holomorphic flat line bundles on E^{an}.

- The universal vector extension of E is the commutative group scheme $E^\natural$ classifying (algebraic) flat line bundles on E.
- Rosenlicht, Grothendieck, Mazur-Messing:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \underline{\Omega^1(E)} & \longrightarrow & E^\natural & \xrightarrow{\pi} & E & \longrightarrow & 0 \\ & & & & [(\mathcal{L},\nabla)] & \longmapsto & [\mathcal{L}] & & \\ & & \omega & \longmapsto & [(\mathcal{O}, d+\omega)] & & & & \end{array}$$

  is universal for extensions of E by vector groups.

- Laumon, Coleman: if k has characteristic zero, then $H^0(E^\natural,\mathcal{O}) = k$ and $H^1(E^\natural,\mathcal{O}) = 0$.

Let $Z \subset E[N](k)$, and set $D := \pi^{-1}(Z) \subset E^\natural$.

Theorem (F.–Matthes; cf. Enriquez–Etingof)

There is a canonical decomposition

$$\Gamma(E^\natural, \Omega^1(\log D)) = \Gamma(E^\natural, \Omega^1) \oplus K^{(1)} \oplus K^{(2)} \oplus \cdots$$

where $K^{(n)}$ are k-subspaces uniquely determined by:

1. $dK^{(n)} = \Gamma(E^\natural, \Omega^1) \wedge K^{(n-1)}$, *where* $K^{(0)} := \pi^*\Gamma(E, \Omega^1)$,
2. $K^{(n)} \wedge K^{(0)} = 0$,
3. $Res_D(K^{(n)})$ *has degree* $n-1$.

We can find $\nu, \omega^{(0)}, \omega_P^{(1)}, \omega_P^{(2)}, \ldots$ such that

$$\Gamma(E^\natural, \Omega^1) = k\nu \oplus k\omega^{(0)}, \qquad K^{(n)} = \bigoplus_{P \in Z} k\omega_P^{(n)}$$

and $d\omega_P^{(n)} = \nu \wedge \omega_P^{(n-1)}$, $\omega_P^{(n)} \wedge \omega^{(0)} = 0$, $Res_D(\omega_P^{(n)}) = t_P^{n-1}/(n-1)!$.

Corollary

Every unipotent flat connection on $E^\natural \setminus D$ is isomorphic to

$$(\mathcal{O} \otimes V, d - \nu \otimes A - \omega^{(0)} \otimes B - \sum_{n \geq 1} \sum_{P \in Z} \omega_P^{(n)} \otimes ad_A^{n-1}(C_P))$$

with $A, B, C_P \in End_k(V)$ "simultaneously nilpotent", and $\sum_{P \in Z} C_P = [A, B]$.

Theorem (F.–Matthes)

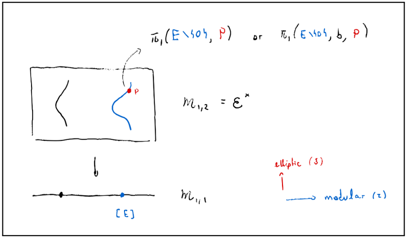
For every $x \in (E^\natural \setminus D)(k)$, there are canonical isomorphism

$$\begin{aligned} \mathcal{O}(\pi_1^{dR}(E^\natural \setminus D, x)) &\cong H^0(B(\Gamma(E^\natural, \Omega^\bullet(\log D))) \\ &\cong T^c \Gamma(E^\natural, \Omega^1(\log D))^{d=0} \end{aligned}$$

Note: if $Q = \pi(x)$, then $\pi_1^{dR}(E^\natural \setminus D, x) \cong \pi_1^{dR}(E \setminus Z, Q)$, and $\Gamma(E^\natural, \Omega^1(\log D))^{d=0} \cong H^1_{dR}(E \setminus Z)$.

Universal elliptic KZB equations

- Differential equations satisfied by multiple elliptic polylogarithms. Calaque–Enriquez–Etingof, Levin–Racinet, Hain, Luo.



9.2. **The formula.** The connection is defined by a 1-form

$$\omega \in \Omega^1(\mathbb{C}\times\mathfrak{h}, \log\Lambda)\hat{\otimes}\,\mathrm{End}\,\mathfrak{p}.$$

via the formula

$$\nabla f = df + \omega f$$

where $f : \mathbb{C}\times\mathfrak{h} \to \mathfrak{p}$ is a (locally defined) section of (9.1). Specifically,

$$\omega = \frac{1}{2\pi i} d\tau \otimes \mathbf{a}\frac{\partial}{\partial \mathbf{t}} + \psi + \nu$$

where

$$\psi = \sum_{m\geq 1}\left(\frac{(2\pi i)^{2m+1}}{(2m)!}G_{2m+2}(\tau)d\tau \otimes \sum_{\substack{j+k=2m+1\\ j,k>0}} (-1)^j [\mathrm{ad}_{\mathbf{t}}^j(\mathbf{a}), \mathrm{ad}_{\mathbf{t}}^k(\mathbf{a})]\frac{\partial}{\partial \mathbf{a}}\right)$$

and

$$\nu = \mathbf{t}F(\xi,\mathbf{t},\tau)\cdot \mathbf{a}\,d\xi + \frac{1}{2\pi i}\left(\frac{1}{\mathbf{t}} + \mathbf{t}\frac{\partial F}{\partial \mathbf{t}}(\xi,\mathbf{t},\tau)\right)\cdot \mathbf{a}\,d\tau.$$

- “Higher level” version: Calaque-Gonzales, Hopper.
- Algebraicity results: Hain, Luo.

Purely algebraic construction of universal KZB:

- Universal vector extension behaves well in families: given $p : E \to S$, get $f : E^\natural \to S$.
- All of the above results generalize to families and are compatible with base change. "Relative fundamental Hopf algebra":

$$\mathcal{H} := H^0(B(f_*\Omega^\bullet_{E^\natural/S}(\log D)).$$

Locally over S:

$$\mathcal{H}^\vee = \mathcal{O}_S\langle\langle a, b, c_P : P \in Z\rangle\rangle / \langle \sum_{P \in Z} c_P - [a, b]\rangle.$$

- Elliptic KZB (vertical direction): integrable S-connection

$$\nabla : f^*\mathcal{H}^\vee \to \Omega^1_{E^\natural/S}(\log D)\hat{\otimes} f^*\mathcal{H}^\vee, \qquad \nabla = d - \Omega$$

$$\Omega = \nu \otimes a + \omega^{(0)} \otimes b + \sum_{n \geq 1} \sum_{P \in Z} \omega_P^{(n)} \otimes ad_a^{n-1}(c_P)$$

- Assume S is a smooth k-scheme.
- Universal KZB (vert. + hor.): integrable k-connection

$$\nabla_{KZB} : f^*\mathcal{H}^\vee \to \Omega^1_{E^\natural/k}(\log D)\hat{\otimes} f^*\mathcal{H}^\vee$$

  lifting ∇. "Isomonodromic deformation".
- Key: the universal vector extension is a crystal. Canonical splitting (with integrability condition) of

$$0 \longrightarrow f^*\Omega^1_{S/k} \longrightarrow \Omega^1_{E^\natural/k} \longrightarrow \Omega^1_{E^\natural/S} \longrightarrow 0$$

  or equivalently of

$$0 \longrightarrow \Omega^1_{S/k} \longrightarrow f_*\Omega^1_{E^\natural/k} \longrightarrow f_*\Omega^1_{E^\natural/S} \longrightarrow 0$$

Theorem (F.–Matthes)

The sequence

$$0 \longrightarrow \Omega^1_{S/k} \longrightarrow f_*\Omega^1_{E^\natural/k}(\log D) \longrightarrow f_*\Omega^1_{E^\natural/S}(\log D) \longrightarrow 0$$

is exact and canonically split.

- Get "canonical lifts" of logarithmic forms. Example:

$$\widetilde{\omega}^{(1)} = \left(\frac{\theta'_\tau(z)}{\theta_\tau(z)} + 2\pi i\, r\right) dz + \frac{1}{2\pi i}\left(\frac{1}{2}\frac{\theta''_\tau(z)}{\theta_\tau(z)} - \frac{1}{6}\frac{\theta'''_\tau(0)}{\theta'_\tau(0)} - \frac{(2\pi i\, r)^2}{2}\right) d\tau$$

- Construct "Gauss–Manin connection for iterated integrals":

$$\nabla_{GM} : \mathcal{H} \to \Omega^1_{S/k} \otimes \mathcal{H}.$$

- $\nabla_{KZB} := f^*\nabla_{GM}^\vee - \widetilde{\Omega}.$