Periods and Poincaré series

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$$2\pi(-1)^{\frac{k}{2}} \left(\frac{n}{m}\right)^{\frac{k-1}{2}} \sum_{\substack{c\geq 1\\ N|c}} \frac{K(-m,n;c)}{c} I_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)$$

- $\mathfrak{H} = \{z \in \mathbb{C} \mid \Im(z) > 0\}$
- $\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \middle| c \equiv 0 \pmod{N} \right\}$

Definition

A *weakly holomorphic modular form of weight k and level N* is a holomorphic function $f : \mathfrak{H} \to \mathbb{C}$ satisfying

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$$

and 'meromorphic at the cusps'. They form a complex vector space $M_k^!(\Gamma_0(N))$.

How to construct?

Definition

Suppose $k \geq 4$ and let $m \in \mathbb{Z}$. The *mth Poincaré series of weight k and level N* is

$$P_{m,k,N}(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \frac{e^{2\pi i m \gamma z}}{(cz+d)^k} \in M_k^!(\Gamma_0(N))$$

where

$$\Gamma_\infty = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \,\middle|\, c = 0 \right\} = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \,\middle|\, n \in \mathbb{Z} \right\}$$

- Can also define Poincaré series in weigth $k = 2$.
- $m = 0 \implies$ Eisenstein series

Every $f \in M_k^!(\Gamma_0(N))$ admits a Fourier series (q-expansion):

$$f(z) = \sum_{n \gg -\infty} a_n(f) q^n, \qquad q = e^{2\pi i z}$$

Example

For $k \geq 4$,

$$P_{0,k,1}(z) = E_k(z) = 1 - \frac{2k}{B_k} \sum_{n \geq 1} \sigma_{k-1}(n) q^n \in M_k(\Gamma_0(1))$$

Example

- $\Delta(z) = \frac{E_4(z)^3 - E_6(z)^2}{1728} = q \prod_{n \geq 1} (1 - q^n)^{24} \in S_{12}(\Gamma_0(1))$
- $j(z) = \frac{E_4(z)^3}{\Delta(z)} = \frac{1}{q} + 744 + 196884q + \cdots \in M_0^!(\Gamma_0(1))$

Definition (Petersson inner product)

For cusp forms $f, g \in S_k(\Gamma_0(N))$, we define

$$(f, g)_{\mathrm{Pet}} = \int_{\Gamma_0(N)\backslash\mathfrak{H}} f(z)\overline{g(z)}y^k \frac{dxdy}{y^2}$$

where $z = x + iy$.

Theorem (Petersson)

For $m \geq 1$, we have $P_{m,k,N} \in S_k(\Gamma_0(N))$ and

$$(f, P_{m,k,N})_{\mathrm{Pet}} = \frac{(k-2)!}{(4\pi m)^{k-1}} a_m(f)$$

for any $f \in S_k(\Gamma_0(N))$.

What about the Fourier coefficients of Poincaré series?

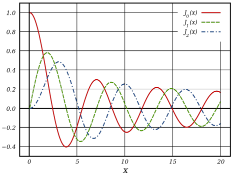
- Are there explicit formulas?
- Are they algebraic, rational, or integers?

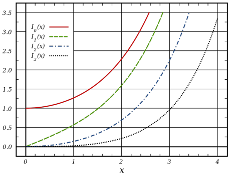
Classical formulas ($m > 0$):

$$P_{m,K,N}(z) = q^m + \sum_{n \geq 1} \left(2\pi(-1)^{\frac{k}{2}} \left(\frac{n}{m}\right)^{\frac{k-1}{2}} \sum_{\substack{c \geq 1 \\ N|c}} \frac{K(m,n;c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right) \right) q^n$$

$$P_{-m,K,N}(z) = q^{-m} + \sum_{n \geq 1} \left(2\pi(-1)^{\frac{k}{2}} \left(\frac{n}{m}\right)^{\frac{k-1}{2}} \sum_{\substack{c \geq 1 \\ N|c}} \frac{K(-m,n;c)}{c} I_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right) \right) q^n$$

- $K(a,b;c) = \sum_{x \in (\mathbb{Z}/c\mathbb{Z})^\times} e^{\frac{2\pi i}{c} ax + bx^{-1}} \in \mathbb{R} \cap \overline{\mathbb{Q}}$ Kloosterman sum
- J_r, I_r Bessel functions





H. POINCARÉ
Fonctions modulaires et fonctions fuchsiennes
Annales de la faculté des sciences de Toulouse 3e série, tome 3 (1911), p. 125-149.

Je me bornerai à constater que $\sum E$ n'est pas nul en général. Il reste à sommer par rapport à γ et notre coefficient s'écrit :

$$\sum_\gamma \mu_j \left[\sum E\right] J\left(m, \frac{4pj\pi^2}{\gamma^2}\right).$$

Il n'y a aucune raison pour qu'il y ait des relations linéaires entre les valeurs des fonctions de Bessel $J\left(m, \frac{4pj\pi^2}{\gamma^2}\right)$ correspondant aux différentes valeurs de γ. Il n'y a donc aucune raison pour que ce coefficient s'annule.

Il en va tout différemment dans le cas de $p = 0$; nos fonctions J se réduisent à une constante simple que je puis faire sortir du signe $\sum$, de sorte que notre coefficient s'écrit :

$$\mu_j J(m, 0) \sum_\gamma \left[\sum E\right].$$

- $P_{1,12,1} = 2.84028... \times \Delta$

((1.5.4) is of course an algebraist's nightmare; one expresses a good integer like $\tau(n)$ as an infinite series with Bessel functions!)

Lehmer's conjecture: $P_{m,12,1} \not\equiv 0$ for every $m \geq 1$.

- $P_{-1,2,1} = -\frac{1}{2\pi i}\frac{dj}{dz} = \frac{1}{q} - 196884q + 42987520q^2 + \cdots$

Corollary: $a_n(j) \sim \frac{e^{4\pi\sqrt{n}}}{\sqrt{2}n^{3/4}}$

- $P_{m,4,8} \equiv 0,\ m = 2, 4, 6, 8, \ldots$

- $P_{-1,4,9} = \frac{1}{q} + 2q^2 - 49q^5 + 48q^8 + 711q^{11} - \cdots \in \mathbb{Z}[\![q]\!]$

(Bruinier-Ono-Rhoades '08, Candelori '14)

- How to explain the rationality phenomena?
- Geometric interpretation for coefficients of Poincaré series?

Periods (à la Kontsevich-Zagier):

Definition. A *period* is a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients, over domains in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients.

Example (Motives $H^n(X)$)

X smooth affine algebraic variety over $\mathbb{Q}$

- $H^n_{\mathrm{dR}}(X) = \Omega^n(X)^{d=0}/d\Omega^{n-1}(X) = \mathbb{Q}\cdot[\omega_1]\oplus\cdots\oplus\mathbb{Q}\cdot[\omega_r]$
- $H^n_{\mathrm{B}}(X) = H_n(X(\mathbb{C});\mathbb{Q})^\vee = \mathbb{Q}\cdot[\sigma_1]^\vee\oplus\cdots\oplus\mathbb{Q}\cdot[\sigma_r]^\vee$
- Comparison isomorphism (Grothendieck '66)

$$\mathrm{comp} : H^n_{\mathrm{dR}}(X)\otimes\mathbb{C} \xrightarrow{\sim} H^n_{\mathrm{B}}(X)\otimes\mathbb{C}$$

- Period matrix

$$P = \begin{pmatrix} \int_{\sigma_1}\omega_1 & \cdots & \int_{\sigma_1}\omega_r \\ \vdots & \ddots & \vdots \\ \int_{\sigma_r}\omega_1 & \cdots & \int_{\sigma_r}\omega_r \end{pmatrix} \in \mathrm{GL}_r(\mathbb{C})$$

Claim

Fourier coefficients of Poincaré series are given by periods of modular motives.

Example

- Elliptic curve $E : y^2 + y = x^3 - x^2 - 10x - 20$
- $H^1_{\mathrm{dR}}(E) = \mathbb{Q} \cdot [\frac{dx}{2y+1}] \oplus \mathbb{Q} \cdot [x\frac{dx}{2y+1}]$, $H^1_{\mathrm{B}}(E) = \mathbb{Q} \cdot [\gamma_1]^\vee \oplus \mathbb{Q} \cdot [\gamma_2]^\vee$
- Period matrix

$$P = \begin{pmatrix} \omega_1 & \eta_1 \\ \omega_2 & \eta_2 \end{pmatrix} = \begin{pmatrix} 1.269... & -2.214... \\ 0.634... + i1.458... & -1.107... + i2.405... \end{pmatrix}$$

- $a_1(P_{1,2,11}) = 1.696... = -\frac{2\pi i}{\omega_1\overline{\omega}_2 - \overline{\omega}_1\omega_2}$
- $a_1(P_{-1,2,11}) = -0.952... = \frac{\overline{\omega}_1\eta_2 - \overline{\omega}_2\eta_1}{\omega_1\overline{\omega}_2 - \overline{\omega}_1\omega_2} - 1$

Single-valued periods (Brown-Dupont '18) of $H^n(X)$ for X smooth affine over $\mathbb{Q}$:

- Combinations of integrals of the form $\int_\sigma \omega \wedge \overline{\eta}$, for $\omega, \eta \in \Omega^n(X)$.
- Complex conjugation $X(\mathbb{C}) \to X(\mathbb{C})$ induces involution $F_\infty : H^n_{\mathrm{B}}(X) \to H^n_{\mathrm{B}}(X)$, which induces

$$\mathrm{sv} : H^n_{\mathrm{dR}}(X) \otimes \mathbb{R} \xrightarrow{\sim} H^n_{\mathrm{dR}}(X) \otimes \mathbb{R}$$

- Single-valued period matrix:

$$S = P^{-1}\overline{P} = P^{-1}F_\infty P \in \mathrm{GL}_r(\mathbb{R})$$

Example $(H^1(E))$

$$S = \frac{1}{2\pi i}\begin{pmatrix} \overline{\omega}_1\eta_2 - \overline{\omega}_2\eta_1 & \overline{\eta}_1\eta_2 - \eta_1\overline{\eta}_2 \\ \omega_1\overline{\omega}_2 - \overline{\omega}_1\omega_2 & \omega_1\overline{\eta}_2 - \omega_2\overline{\eta}_1 \end{pmatrix} = \begin{pmatrix} -0.028\ldots & -1.695\ldots \\ -0.589\ldots & 0.028\ldots \end{pmatrix}$$

Given a level $N \geq 1$, we have a modular curve $Y_0(N)$ over $\mathbb{Q}$ such that

$$Y_0(N)(\mathbb{C}) = \Gamma_0(N) \backslash \mathfrak{H}$$

To a weight $k \geq 2$ and a level $N \geq 1$ we associate a motive

$$M(k,N) = H^1_{\mathrm{cusp}}(\mathcal{Y}_0(N), \mathrm{Sym}^{k-2} H^1(\mathcal{E}/\mathcal{Y}_0(N)))$$

subquotient of

$$H^{k-1}(\underbrace{\mathcal{E} \times_{\mathcal{Y}_0(N)} \cdots \times_{\mathcal{Y}_0(N)} \mathcal{E}}_{k-2})$$

Example

- Let $X_0(N) = \overline{Y_0(N)}$. Then $M(2,N) = H^1(X_0(N))$.
- In the example before, $X_0(11) = E$. Fourier coefficients of $P_{m,2,11}$ are single-valued periods of $M(2,11)$.

Theorem

Let $S = (s_{ij})_{1 \le i,j \le r} \in \mathrm{GL}_r(\mathbb{R})$ *be a single-valued period matrix with respect to a* $\mathbb{Q}$*-basis of* $M(k,N)_{\mathrm{dR}}$*. Then,*

$$\mathbb{Q}(s_{ij} : 1 \le i,j \le r) = \mathbb{Q}(a_n(P_{m,k,N}) : m,n \in \mathbb{Z}).$$

- $\mathbb{Q}(a_n(P_{m,k,N}) : m,n \in \mathbb{Z})$ is finitely generated.
- If $M(k,N) = 0$, then $a_n(P_{m,k,N}) \in \mathbb{Q}$ for every m,n.

  Example: $M(2,1) = H^1(X_0(1)) = H^1(\mathbb{P}^1) = 0$.

$$P_{-1,2,1} = \frac{1}{q} - 196884q + 42987520q^2 + \cdots$$

Let $D = \frac{1}{2\pi i}\frac{d}{dz} = q\frac{d}{dq}$.

Theorem (Scholl '85, Coleman '96, Brown-Hain '18, ...)

For every $k \geq 2$, $N \geq 1$, there is a canonical isomorphism

$$M(k,N)_{\mathrm{dR}} \cong S_k^!(\Gamma_0(N))_{\mathbb{Q}}/D^{k-1}M_{2-k}^!(\Gamma_0(N))_{\mathbb{Q}}$$

[$f \in S_k^!$ if constant term at the cusps vanish, e.g. $a_0(f) = 0$]

Example ($k = 2$)

- We have $M_2^!(\Gamma_0(N))_{\mathbb{Q}} \cong \Omega^1(Y_0(N))$ via $f \mapsto 2\pi i\, f(z)dz$, so that

  $$M_2^!(\Gamma_0(N))_{\mathbb{Q}}/DM_0^!(\Gamma_0(N))_{\mathbb{Q}} \cong \Omega^1(Y_0(N))/d\mathcal{O}(Y_0(N)) = H^1_{\mathrm{dR}}(Y_0(N))$$

- $S_2^!(\Gamma_0(N))_{\mathbb{Q}}$: 1-forms with vanishing residues along the cusps

  $$S_2^!(\Gamma_0(N))_{\mathbb{Q}}/DM_0^!(\Gamma_0(N))_{\mathbb{Q}} \cong H^1_{\mathrm{dR}}(X_0(N)) = M(2,N)_{\mathrm{dR}}$$

- Assume $M(k, N)$ has rank 2.
- Let $f \in S_k(\Gamma_0(N))_{\mathbb{Q}}$ and $g \in S_k^!(\Gamma_0(N))_{\mathbb{Q}}$ induce a basis of $M(k, N)_{\mathrm{dR}}$.
- Let $S = \left(\begin{smallmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{smallmatrix}\right)$ be the corresponding single-valued period matrix.

Theorem

For every $m \geq 1$, there is $h_m \in M_{2-k}^!(\Gamma_0(N))_{\mathbb{Q}}$ such that, for every $n \geq 1$,

$$a_n(P_{m,k,N}) = -\frac{(k-2)!}{m^{k-1}} a_m(f) a_n(f) \frac{1}{s_{21}}$$

$$a_n(P_{-m,k,N}) = \frac{(k-2)!}{m^{k-1}} a_m(f) a_n(f) \frac{s_{11}}{s_{21}} + r_{m,n}$$

where $r_{m,n} = \frac{(k-2)!}{m^{k-1}} a_m(f) a_n(g) + n^{k-1} a_n(h_m) \in \mathbb{Q}$.

Complex multiplications by $L = \mathbb{Q}(\sqrt{-d})$:

- Assume $M(k, N) \otimes L$ admits a non-trivial endomorphism.
- We get $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in M_{r\times r}(L)$ such that

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \begin{pmatrix} \overline{a} & \overline{b} \\ 0 & \overline{d} \end{pmatrix}$$

- Thus:

$$\frac{s_{11}}{s_{21}} = \frac{b}{\overline{a} - a} \in L \cap \mathbb{R} = \mathbb{Q}$$

Example

$M(4, 9)$ has CM by $\mathbb{Q}(\sqrt{-3})$. Corollary: $P_{-m,4,9}$ has rational Fourier coefficients for every $m \geq 1$.

How to prove the theorems?

- Explicit description of

$$\mathrm{sv} : M(k,N)_{\mathrm{dR}} \otimes \mathbb{R} \to M(k,N)_{\mathrm{dR}} \otimes \mathbb{R}$$

  via harmonic Maass forms:

$$\mathrm{sv}([f]) = \frac{(4\pi)^{k-1}}{(k-2)!}[D^{k-1}(F)]$$

  where $F \in H^!_{2-k}(\Gamma_0(N))$ is a *harmonic lift* of f:

$$\frac{2i}{(\Im z)^{2-k}} \overline{\frac{\partial F}{\partial \overline{z}}} = f(z)$$

- Bringmann-Ono '07 $\implies \mathrm{sv}([P_{m,k,N}]) = -[P_{-m,k,N}]$.

Some references:

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