Higher Ramanujan Foliations

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Halphen:

$$\frac{d(u_1+u_2)}{dt}=u_1u_2,\qquad \frac{d(u_2+u_3)}{dt}=u_2u_3,\qquad \frac{d(u_3+u_1)}{dt}=u_3u_1$$

Chazy:

$$y'''=2y'y''-3(y')^2$$

Ramanujan:

$$q\frac{dE_2}{dq}=\frac{E_2^2-E_4}{12},\qquad q\frac{dE_4}{dq}=\frac{E_2E_4-E_6}{3},\qquad q\frac{dE_6}{dq}=\frac{E_2E_6-E_4^2}{2}$$

$$y=u_1+u_2+u_3=E_2,\qquad q=e^{2t}=e^{2\pi i\tau}$$

- Define an open $U \subset \mathrm{SL}_2(\mathbb{C})$ by

$$\begin{array}{ccccc} U & \hookrightarrow & \mathrm{SL}_2(\mathbb{C}) & & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ \downarrow & \square & \downarrow & & \downarrow \\ \mathbb{H} & \hookrightarrow & \mathbb{P}^1(\mathbb{C}) & & (b:d) \\ \tau & \longmapsto & (\tau:1) & & \end{array}$$

  and set

$$X = \mathrm{SL}_2(\mathbb{Z}) \backslash U$$

- Let v be the vector field on X induced by

$$\frac{1}{2\pi i} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{C})$$

Theorem (uniformization)

(X, v) *is biholomorphic to*

$$\left(\mathbb{C}^3 \setminus V(x_2^3 - x_3^2),\ \frac{x_1^2 - x_2}{12}\frac{\partial}{\partial x_1} + \frac{x_1 x_2 - x_3}{3}\frac{\partial}{\partial x_2} + \frac{x_1 x_3 - x_2^2}{2}\frac{\partial}{\partial x_3}\right)$$

Proof.

Consider the holomorphic map

$$U \longrightarrow \mathbb{C}^3$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \left(\frac{1}{d^2}E_2(\tau) - \frac{12}{2\pi i}\frac{c}{d},\ \frac{1}{d^4}E_4(\tau),\ \frac{1}{d^6}E_6(\tau)\right)$$

where $\tau = b/d$ and $E_2, E_4, E_6 : \mathbb{H} \to \mathbb{C}$ are normalized Eisenstein series. $\square$

Moduli point of view (Movasati)

E complex elliptic curve $\rightsquigarrow$ $H^1_{\mathrm{dR}}(E)$ 2-dim $\mathbb{C}$-vector space with:

- symplectic pairing $\langle\ ,\rangle : H^1_{\mathrm{dR}}(E) \times H^1_{\mathrm{dR}}(E) \to \mathbb{C}$
- 1-dim subspace $H^0(E, \Omega^1) \subset H^1_{\mathrm{dR}}(E)$

Symplectic-Hodge basis of $H^1_{\mathrm{dR}}(E)$: $b = (\omega, \eta)$ such that

$$\omega \in H^0(E, \Omega^1) \ \text{ and } \ \langle \omega, \eta \rangle = 1$$

Theorem

X is biholomorphic to the moduli space of elliptic curves with a SH basis (E, b) and v gets identified with the unique vector field satisfying

$$\nabla_v \omega^{univ} = \eta^{univ} \ \text{ and } \ \nabla_v \eta^{univ} = 0$$

where ∇ denotes the Gauss-Manin connection.

Gauss-Manin connection:

$$\int_\gamma \nabla\alpha = d\int_\gamma \alpha$$

Proof of the theorem.

Consider

$$[(E,b)] \longmapsto \left[\begin{pmatrix} \int_{\gamma_2}\eta & \frac{1}{2\pi i}\int_{\gamma_2}\omega \\ \int_{\gamma_1}\eta & \frac{1}{2\pi i}\int_{\gamma_1}\omega \end{pmatrix}\right] \in \mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{C})$$

where (γ_1,γ_2) is a oriented basis of $H_1(E,\mathbb{Z})$. □

Remark

The identification of the moduli space with $\mathbb{A}^3 \setminus V(x_2^3 - x_3^2)$ is actually purely algebraic. Also works over $\mathbb{Q}$, or even $\mathbb{Z}[1/6]$.

(A, λ) principally polarized abelian variety of dimension $g \rightsquigarrow$ $H^1_{\mathrm{dR}}(A)$ $2g$-dim vector space with:

- symplectic pairing $\langle \ , \rangle_\lambda$ on $H^1_{\mathrm{dR}}(A)$
- Lagrangian subspace $H^0(A, \Omega^1) \subset H^1_{\mathrm{dR}}(A)$

Symplectic-Hodge basis of $H^1_{\mathrm{dR}}(A)$: $b = (\omega_1, \ldots, \omega_g, \eta_1, \ldots, \eta_g)$ such that

$$\omega_i \in H^0(A, \Omega^1) \text{ and } b \text{ is symplectic wrt } \langle \ , \rangle_\lambda$$

Theorem

The moduli problem of principally polarized abelian varieties of dimension g with a SH basis is representable by a smooth quasi-affine variety B_g of dimension $2g^2 + g$.

Definition

The higher Ramanujan foliation on B_g is the rank $g(g+1)/2$ subbundle $\mathcal{R}_g \subset TB_g$ of the vector fields v such that $\nabla_v \eta_i^{univ} = 0$ for every i.

- There is a canonical holomorphic map tangent to $\mathcal{R}_g$ generalizing (E_2, E_4, E_6):

  $$\varphi_g : \mathbb{H}_g \longrightarrow B_g$$

  Its "coordinates" are "Siegel quasimodular forms".
- Classical result: E_2, E_4, E_6 are algebraically independent.

Theorem

Every analytic leaf of $\mathcal{R}_g$ is Zariski-dense in B_g.

Theorem (Nesterenko '96)

π, e^π, and $\Gamma(1/4)$ are algebraically independent over $\mathbb{Q}$.

Proof: interpret as values of Eisenstein series and use Ramanujan's equations.

Conjecture

Three at least of the four numbers

$$\pi,\ e^{\pi\sqrt{5}},\ \Gamma(1/5),\ \Gamma(2/5)$$

are algebraically independent over $\mathbb{Q}$.

Idea: approach it through φ_g and $\mathcal{R}_g$?

- T. J. Fonseca, *Higher Ramanujan Equations and Periods of Abelian Varieties*, to appear in Memoirs of the AMS.

- J. Cao, H. Movasati, S-T. Yau, *Gauss-Manin Connection in Disguise: Genus Two Curves*, preprint 2019.