From transcendental numbers to higher Ramanujan equations

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Seminário Fique em Casa de Geometria Algébrica
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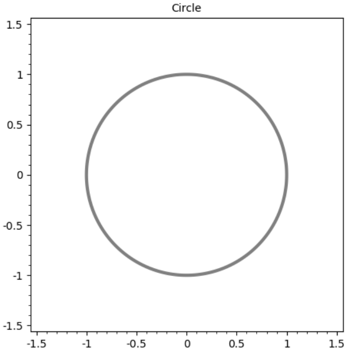
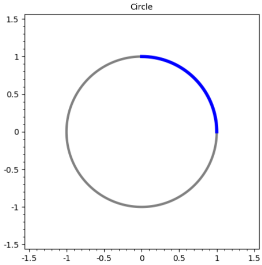


$$x^2 + y^2 = 1$$



$$x^2 + y^2 = 1$$

$$\frac{\pi}{2} = \int_0^1 \frac{1}{\sqrt{1 - x^2}} dx$$

Theorem (Lindemann 1882)

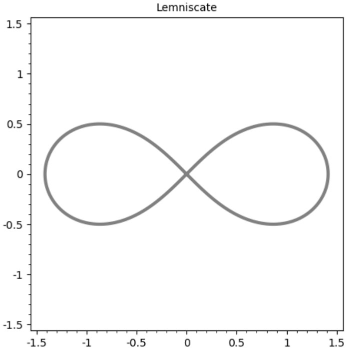
π is transcendental.

Ueber die Zahl π.*)

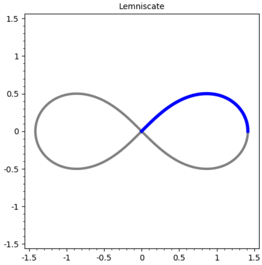
Von

F. Lindemann in Freiburg i. Br.

Bei der Vergeblichkeit der so ausserordentlich zahlreichen Versuche**), die Quadratur des Kreises mit Cirkel und Lineal auszuführen, hält man allgemein die Lösung der bezeichneten Aufgabe für unmöglich; es fehlte aber bisher ein Beweis dieser Unmöglichkeit; nur die Irrationalität von π und von π^2 ist festgestellt. Jede mit Cirkel und Lineal ausführbare Construction lässt sich mittelst algebraischer Einkleidung zurückführen auf die Lösung von linearen und quadratischen Gleichungen, also auch auf die Lösung einer Reihe von quadratischen Gleichungen, deren erste rationale Zahlen zu Coefficienten hat, während die Coefficienten jeder folgenden nur solche irrationale Zahlen enthalten, die durch Auflösung der vorhergehenden Gleichungen eingeführt sind. Die Schlussgleichung wird also durch wiederholtes Quadriren übergeführt werden können in eine Gleichung geraden Grades, deren Coefficienten rationale Zahlen sind. Man wird sonach die Unmöglichkeit der Quadratur des Kreises darthun, wenn man nachweist, dass *die Zahl π überhaupt nicht Wurzel einer algebraischen Gleichung irgend welchen Grades mit rationalen Coefficienten sein kann.* Den dafür nöthigen Beweis zu erbringen, ist im Folgenden versucht worden.



$$(x^2 + y^2)^2 - 2(x^2 - y^2) = 1$$



Theorem (Schneider 1937)

Ω is transcendental.

$$(x^2+y^2)^2-2(x^2-y^2)=1$$

$$\frac{\Omega}{2}=\int_0^1 \frac{1}{\sqrt{1-r^4}}dr=\frac{\Gamma(1/4)^2}{4\sqrt{2\pi}}$$

Arithmetische Untersuchungen elliptischer Integrale.

Von

Theodor Schneider in Frankfurt a. M.

Im Jahre 1931 zeigte C. L. Siegel[1]), daß die Größen g_2, g_3, ω_1, ω_2 nicht sämtlich algebraisch sind, wobei ω_1 und ω_2 die Perioden der durch g_2 und g_3 bestimmten Weierstraßschen $\wp$-Funktion seien. Mit einer ähnlichen Methode bewies ich im Februar 1934[2]), daß auch g_2, g_3, ω, η nicht alle algebraisch sind, wenn ω und η demselben Integrationsweg entsprechen. Im zweiten Teil meiner Dissertation[3]) führte ich den Beweis des Satzes: Besteht zwischen β und $\tau=\frac{\omega_2}{\omega_1}$ keine lineare Beziehung mit rationalen Koeffizienten, so ist mindestens eine der Größen g_2, g_3, τ, β und $\wp(\omega_1\beta)$ transzendent. Auf Grund anderer Beweisprinzipien gelangte im Vorjahre G. Pólya[4]) zu der Aussage: Wenn h eine komplexe Zahl vom Absolutbetrage < 1 ist, so sind nicht alle fünf Zahlen

$$\sum_r \frac{r\,h^{2r}}{1-h^{2r}},\quad \sum_r \frac{r^3\,h^{2r}}{1-h^{2r}},\quad \sum_r \frac{r^5\,h^{2r}}{1-h^{2r}},$$

$$\frac{\sum_r(-1)^r(2r-1)\,h^{(2r-1)^2}}{\prod_r(1-h^{8r})(1-h^{8r-4})^2},\quad \frac{\sum_r(-1)^r r^2\,h^{4r^2}}{\prod_r(1-h^{8r})(1-h^{8r-4})^2}\qquad (r=1,2,3,\ldots)$$

algebraisch. Diese ist wieder enthalten in dem vor einigen Monaten von P. Popken und K. Mahler[5]) bewiesenen Satz, daß auch die drei Größen $\frac{\omega\eta}{\pi^2}$, $\frac{\omega^4 g_2}{\pi^4}$, $\frac{\omega^6 g_3}{\pi^6}$ nicht sämtlich algebraisch sind. Alle vorstehenden Einzel-

Is Ω, or $\Gamma(1/4)$, algebraically independent to π?

Recall: $\alpha, \beta \in \mathbb{C}$ are algebraically independent (over $\mathbb{Q}$) if there exists no non-zero $P \in \mathbb{Q}[X, Y]$ such that $P(\alpha, \beta) = 0$.

Theorem (Chudnovsky 1976)

π and $\Gamma(1/4)$ are algebraically independent.

Also true for π and $\Gamma(1/3)$.

Algebraic Independence of Values of Exponential and Elliptic Functions

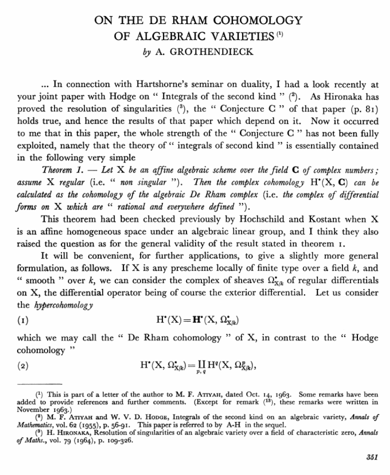
G. V. Chudnovsky

0. Only one Congress separates us from 1982, the centenary of Lindemann's theorem on the transcendence of π. Many things have changed since 1882 in Transcendence Theory. For the last years especially there has been considerable progress in understanding the fundamental problems of Transcendence Theory. Although the analytic part of proofs looks like 40 years ago the algebraic arguments have changed completely. Now Transcendence Theory uses a lot of modern mathematics (algebra, algebraic geometry, complex analysis) and also has its fields of application. We'll try to describe the new situation with the theory of transcendence and algebraic independence for the exponential, elliptic and Abelian functions.

Let $\wp(z)$ denote the Weierstrass elliptic function with algebraic invariants g_2, g_3 and $\zeta(z)$ the ζ-function, $\zeta'(z) = -\wp(z)$. Let ω, η denote any pair of periods and quasi-periods of $\wp(z)$: $\zeta(z+\omega) = \zeta(z) + \eta$, and let ω_i, η_i denote fundamental periods and quasi-periods of $\wp(z)$. We call point u as algebraic for $\wp(z)$ if $\wp(u) \in \overline{Q}$. For a finite set $S \subset C$, $\# S$ denotes the maximal number of algebraically independent (a.i.) elements in S.

Open problem: are $\pi, \Gamma(1/5), \Gamma(2/5)$ algebraically independent?

- A. Grothendieck, *On the de Rham cohomology of algebraic varieties*. Publications Mathématiques de l'IHES, tome 29 (1966)



ON THE DE RHAM COHOMOLOGY
OF ALGEBRAIC VARIETIES [1]
by A. GROTHENDIECK

... In connection with Hartshorne's seminar on duality, I had a look recently at your joint paper with Hodge on " Integrals of the second kind " [2]. As Hironaka has proved the resolution of singularities [3], the " Conjecture C " of that paper (p. 81) holds true, and hence the results of that paper which depend on it. Now it occurred to me that in this paper, the whole strength of the " Conjecture C " has not been fully exploited, namely that the theory of " integrals of second kind " is essentially contained in the following very simple

Theorem 1. — Let X *be an affine algebraic scheme over the field* **C** *of complex numbers; assume* X *regular* (i.e. " *non singular* "). *Then the complex cohomology* $H^\bullet(X, \mathbf{C})$ *can be calculated as the cohomology of the algebraic De Rham complex* (i.e. *the complex of differential forms on* X *which are* " *rational and everywhere defined* ").

This theorem had been checked previously by Hochschild and Kostant when X is an affine homogeneous space under an algebraic linear group, and I think they also raised the question as for the general validity of the result stated in theorem 1.

It will be convenient, for further applications, to give a slightly more general formulation, as follows. If X is any prescheme locally of finite type over a field k, and " smooth " over k, we can consider the complex of sheaves $\Omega^\bullet_{X/k}$ of regular differentials on X, the differential operator being of course the exterior differential. Let us consider the *hypercohomology*

(1) $$H^\bullet(X) = \mathbf{H}^\bullet(X, \Omega^\bullet_{X/k})$$

which we may call the " De Rham cohomology " of X, in contrast to the " Hodge cohomology "

(2) $$H^\bullet(X, \Omega^\bullet_{X/k}) = \coprod_{p,q} H^q(X, \Omega^p_{X/k}),$$

(1) This is part of a letter of the author to M. F. Atiyah, dated Oct. 14, 1963. Some remarks have been added to provide references and further comments. (Except for remark (13), these remarks were written in November 1963.)
(2) M. F. Atiyah and W. V. D. Hodge, Integrals of the second kind on an algebraic variety, *Annals of Mathematics*, vol. 62 (1955), p. 56-91. This paper is referred to by A-H in the sequel.
(3) H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero, *Annals of Maths.*, vol. 79 (1964), p. 109-326.

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$X_{/\mathbb{Q}}$ smooth algebraic variety

- $H^n_{\mathrm{dR}}(X) = \mathbb{H}^n(\Omega^\bullet_{X/\mathbb{Q}})$
- $H^n_{\mathrm{B}}(X) = H_n(X(\mathbb{C}), \mathbb{Q})^\vee$

$$\mathrm{comp} : H^n_{\mathrm{dR}}(X) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H^n_{\mathrm{B}}(X) \otimes_{\mathbb{Q}} \mathbb{C}$$
$$[\alpha] \mapsto ([\gamma] \mapsto \int_\gamma \alpha)$$

Period matrix:

$$P = (\int_{\gamma_j} \alpha_i)_{i,j}$$

$[\alpha_i]$ and $[\gamma_j]$ bases defined over $\mathbb{Q}$

Example (Elliptic curves)

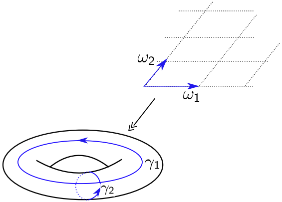
Let $E \subset \mathbb{P}^2$ be given by $y^2z = 4x^3 - uxz^2 - vz^3$, with $u, v \in \mathbb{Q}$, satisfying $u^3 - 27v^2 \neq 0$.
From

$$H^1_{\mathrm{dR}}(E) = H^1_{\mathrm{dR}}(E^{\mathrm{aff}}), \qquad E^{\mathrm{aff}} = E \setminus \{O\}$$

we obtain

$$H^1_{\mathrm{dR}}(E) = \mathbb{Q} \cdot \underbrace{\left[\frac{dx}{y}\right]}_{\omega} \oplus \mathbb{Q} \cdot \underbrace{\left[x\frac{dx}{y}\right]}_{\eta}$$

$$P = \begin{pmatrix} \omega_1 & \eta_1 \\ \omega_2 & \eta_2 \end{pmatrix} = \begin{pmatrix} \int_{\gamma_1}\omega & \int_{\gamma_1}\eta \\ \int_{\gamma_2}\omega & \int_{\gamma_2}\eta \end{pmatrix}$$



- Legendre: $\omega_1\eta_2 - \omega_2\eta_1 = 2\pi i$
- In the case $u = 4, v = 0$, we have $\omega_1 = \Omega$, $\omega_2 = i\Omega$

Crucial remark: algebraic cycles induce relations between periods!

Example (Complex multiplication)

Let $\varphi \in \mathrm{End}(E_{\mathbb{C}}) \setminus \mathbb{Z}$ (corresponds to 1-cycle in $E \times E$).
From

$$\varphi_{\mathrm{B}}^{*} \circ \mathrm{comp} = \mathrm{comp} \circ \varphi_{\mathrm{dR}}^{*}$$

$$\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{\in \mathrm{GL}_2(\mathbb{Q})} \begin{pmatrix} \omega_1 & \eta_1 \\ \omega_2 & \eta_2 \end{pmatrix} = \begin{pmatrix} \omega_1 & \eta_1 \\ \omega_2 & \eta_2 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}$$

we get

$$\frac{a\omega_1 + b\omega_2}{c\omega_1 + d\omega_2} = \frac{\omega_1}{\omega_2} \Longrightarrow \frac{\omega_1}{\omega_2} \in \overline{\mathbb{Q}}$$

Conjecture (Grothendieck)

Every algebraic relation between periods is of motivic origin.

([10]) In fact, J.-P. Serre pointed out to me that for an algebraic curve over $\mathbf{C}$, these " periods of differentials of the second kind " are rather classical invariants. Thus, for an elliptic curve defined by the periods ω_1, ω_2 one defines classically the integrals

$$\eta_i = \int_0^{\omega_i} \eta,$$

(where $x = \wp z$, $y = \wp' z$, and $\eta = \frac{x\,dx}{y}$ is a differential of the second kind which, together with the invariant differential ω, forms a basis of $H^1(X)$ = differentials of second kind mod. exact differentials). The only known general algebraic relation among the η_i and ω_i is

$$\omega_1 \eta_2 - \eta_1 \omega_2 = 2i\pi.$$

Schneider's theorem states that if X is algebraic (i.e. its coefficients g_2 and g_3 are algebraic), then ω_1 and ω_2 are transcendental, and it is believed that if X has no complex multiplication, then ω_1 and ω_2 are algebraically independent. This conjecture extends in an obvious way to the set of periods $(\omega_1, \omega_2, \eta_1, \eta_2)$ and can be rephrased also for curves of any genus, or rather for abelian varieties of dimension g, involving $4g$ periods.

Can rephrase as $(\pm\epsilon)$:

$$\mathrm{trdeg}_{\mathbb{Q}} \mathbb{Q}(\mathrm{Periods}(X)) = \dim G_{mot}(X)$$

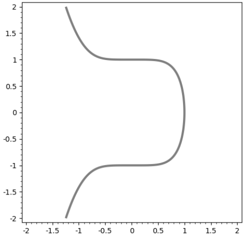
Example (Elliptic curve)

$$\dim G_{mot}(E) = \begin{cases} 2 & \text{if } E \text{ has CM} \\ 4 & \text{if not} \end{cases}$$

Example (A genus 2 curve)

- $C_{/\mathbb{Q}}$ = hyperelliptic curve with affine equation $y^2 = 1 - x^5$. Automorphism

$$\sigma : (x, y) \mapsto (\zeta x, y), \qquad \zeta = e^{\frac{2\pi i}{5}}$$



- For suitable γ_j $(1 \le i, j \le 4)$:

$$\int_{\gamma_j} x^{i-1} \frac{dx}{y} = \frac{2}{5} \zeta^{i(j-1)} (1 - \zeta^i) B\left(\frac{i}{5}, \frac{1}{2}\right)$$

Get:

$$\mathbb{Q}(\text{Periods}(C)) \overset{alg}{\subset} \overline{\mathbb{Q}}(\pi, \Gamma(1/5), \Gamma(2/5))$$

Can prove $\dim G_{mot}(C) = 3$, so the period conjecture predicts $\pi, \Gamma(1/5), \Gamma(2/5)$ algebraically independent.

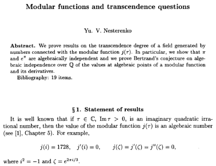
Chudnovsky's method only gives "at least two of $\pi, \Gamma(1/5), \Gamma(2/5)$ are algebraically independent"...

Is there another approach?

Theorem (Nesterenko 1996)

For every $\tau \in \mathbb{H} = \{z \in \mathbb{C} \mid \Im z > 0\}$,

$$trdeg_{\mathbb{Q}} \mathbb{Q}(e^{2\pi i\tau}, E_2(\tau), E_4(\tau), E_6(\tau)) \geq 3$$



Modular functions and transcendence questions

Yu. V. Nesterenko

Abstract. We prove results on the transcendence degree of a field generated by numbers connected with the modular function $j(\tau)$. In particular, we show that π and e^π are algebraically independent and we prove Bertrand's conjecture on algebraic independence over $\mathbb{Q}$ of the values at algebraic points of a modular function and its derivatives.

Bibliography: 19 items.

§1. Statement of results

It is well known that if $\tau \in \mathbb{C}$, $\operatorname{Im}\tau > 0$, is an imaginary quadratic irrational number, then the value of the modular function $j(\tau)$ is an algebraic number (see [1], Chapter 5). For example,

$$j(i) = 1728, \quad j'(i) = 0, \qquad j(\zeta) = j'(\zeta) = j''(\zeta) = 0,$$

where $i^2 = -1$ and $\zeta = e^{2\pi i/3}$.

- $E_{2k}(\tau) = 1 - \frac{4k}{B_{2k}} \sum_{n\geq 1} \sigma_{2k-1}(n) q^n \in \mathbb{Z}[\![q]\!]$, where $q = e^{2\pi i\tau}$
- Improves Chudnovsky! Given $E : y^2 = 4x^3 - ux - v$,

$$E_2(\tau) = 12\frac{\omega_1\eta_1}{(2\pi i)^2}, \qquad E_4(\tau) = 12u\left(\frac{\omega_1}{2\pi i}\right)^4, \qquad E_6(\tau) = -216v\left(\frac{\omega_1}{2\pi i}\right)^6$$

  where $\tau = \omega_2/\omega_1$

- E.g. $e^\pi, \pi, \Gamma(1/4)$ are algebraically independent
- Proof relies on integrality plus Ramanujan's equations

$$DE_2 = \frac{E_2^2 - E_4}{12}, \qquad DE_4 = \frac{E_2E_4 - E_6}{3}, \qquad DE_6 = \frac{E_2E_6 - E_4^2}{2}$$

  where $D = \frac{1}{2\pi i}\frac{d}{d\tau} = q\frac{d}{dq}$

Several variables generalization ?

Zudilin 2000

Sbornik: Mathematics **191**:12 000–000
Matematicheskiĭ Sbornik **191**:12 77–122
UDC 511.334, 517.953

Thetanulls and differential equations

V. V. Zudilin

Abstract. The closedness of the system of thetanulls (and the Siegel modular forms) and their first derivatives with respect to differentiation is well-known in the one-dimensional case. It is shown in the present paper that thetanulls and their various logarithmic derivatives satisfy a non-linear system of differential equations; only one and two-dimensional versions of this result were known before. Several distinct examples of such systems are presented, and a theorem on the transcendence degree of the differential closure of the field generated by all thetanulls is established. On the basis of a study of the modular properties of logarithmic derivatives of thetanulls (previously unknown) relations between these functions and thetanulls themselves are obtained in dimensions 2 and 3.

Bibliography: 26 titles.

Introduction

Theta functions is a classical domain of mathematics, marked with beauty. Many results in mathematical analysis, algebraic geometry, differential equations, and other areas owe their existence to its development. There are quite a few monographs (sound examples are [1]–[3]) and papers dedicated to theta functions, of which a small part can be found in our list of literature.

The first systematic study of one-dimensional theta functions was carried out by Jacobi (see [4], [5]), although his notation is distinct from the following notation, which was used in later papers of Frobenius, Krazer, Wirtinger, and other authors:

$$\vartheta_1(z,q) = 2\sum_{n=0}^{\infty}(-1)^n q^{(n+1/2)^2}\sin(2n+1)z = 2q^{1/4}\sin z - 2q^{9/4}\sin 3z + 2q^{25/4}\sin 5z - \cdots,$$

$$\vartheta_2(z,q) = 2\sum_{n=0}^{\infty} q^{(n+1/2)^2}\cos(2n+1)z = 2q^{1/4}\cos z + 2q^{9/4}\cos 3z + 2q^{25/4}\cos 5z + \cdots,$$

AMS 1991 Mathematics Subject Classification. Primary 14K25, 11F46; Secondary 35Rxx.

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Pellarin 2004

Introduction aux formes modulaires de Hilbert et à leur propriétés différentielles.

Federico Pellarin

Table des matières

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Geometric approach (cf. Movasati's "Gauss-Manin in disguise")

- k base field
- $(A, \lambda)_{/k}$ principally polarized abelian variety of dimension g
- $H^1_{\mathrm{dR}}(A)$ is a $2g$-dimensional k-vector space with:
  1. a symplectic k-form $\langle \ , \ \rangle_\lambda : H^1_{\mathrm{dR}}(A) \times H^1_{\mathrm{dR}}(A) \to k$
  2. a Lagrangian subspace $H^0(A, \Omega^1) \subset H^1_{\mathrm{dR}}(A)$ (Hodge filtration)
- Symplectic-Hodge basis: $b = (\omega_1, \ldots, \omega_g, \eta_1, \ldots, \eta_g)$ such that

$$\omega_i \in H^0(A, \Omega^1) \quad \text{and} \quad b \text{ is symplectic wrt } \langle \ , \ \rangle_\lambda$$

Example (g=1)

$([dx/y], [xdx/y])$ is a symplectic Hodge basis of an elliptic curve given by $y^2 = 4x^3 - ux - v$

Theorem

There is a smooth Deligne-Mumford stack $\mathcal{B}_g$ over $\mathbb{Z}$ classifying (A, λ, b). The base change $\mathcal{B}_g \otimes \mathbb{Z}[1/2]$ is representable by a smooth quasi-affine $\mathbb{Z}[1/2]$-scheme B_g of rel. dimension $2g^2 + g$.

Example ($g = 1$)

$B_1 \otimes \mathbb{Z}[1/6] \cong \operatorname{Spec} \mathbb{Z}[1/6, x_1, x_2, x_3, (x_2^3 - x_3^2)^{-1}]$

We can see

$$\begin{array}{ccc} (A, \lambda, b) & & \mathcal{B}_g \\ \downarrow & & \downarrow \pi \\ (A, \lambda) & & \mathcal{A}_g \end{array}$$

as a principal P_g-bundle

$$P_g = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \leq \mathsf{Sp}_{2g}$$

Claim: there is a canonical splitting of

$$0 \longrightarrow T_{\mathcal{B}_g/\mathcal{A}_g} \longrightarrow T_{\mathcal{B}_g} \xrightarrow{D\pi} \pi^* T_{\mathcal{A}_g} \longrightarrow 0$$

Consider the vector bundle

$$\begin{array}{ccc} \mathcal{V} & & H^1_{\mathrm{dR}}(A)^{\oplus g} \\ \downarrow p & & \downarrow \\ \mathcal{A}_g & & (A, \lambda) \end{array}$$

We have an immersion over $\mathcal{A}_g$:

$$\begin{array}{ccc} (A, \lambda, b) & \longmapsto & (A, \lambda, \eta_1, \dots, \eta_g) \\ \mathcal{B}_g & \xhookrightarrow{i} & \mathcal{V} \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle p} & \\ & \mathcal{A}_g & \end{array}$$

The Gauss-Manin connection induces a splitting (Ehresmann)

$$0 \to T_{\mathcal{V}/\mathcal{A}_g} \overset{\leftarrow --}{\to} T_{\mathcal{V}} \xrightarrow{Dp} p^* T_{\mathcal{A}_g} \to 0$$

This induces a splitting of the original sequence via i.

We get an (integrable) subbundle $\mathcal{R}_g \subset T_{\mathcal{B}_g}$ isomorphic to $\pi^* T_{\mathcal{A}_g}$

- Let $\mathcal{F}$ = v. bun. over $\mathcal{A}_g$ whose fiber at (A, λ) is $H^0(A, \Omega^1)$.
- Kodaira-Spencer:
$$T_{\mathcal{A}_g} \cong \mathrm{Sym}^2(\mathcal{F})^\vee$$
- $\pi^*\mathcal{F}$ trivialized by $(\omega_1^{univ}, \ldots, \omega_g^{univ})$, so we get a trivialization
$$(v_{ij})_{1 \leq i \leq j \leq g}$$
of $\mathcal{R}_g \cong \pi^* T_{\mathcal{A}_g}$, the higher Ramanujan vector fields.

Example ($g = 1$)

Under the previous identification of $B_1 \otimes \mathbb{Z}[1/6]$, we get

$$v_{11} = \frac{x_1^2 - x_2}{12}\frac{\partial}{\partial x_1} + \frac{x_1 x_2 - x_3}{3}\frac{\partial}{\partial x_2} + \frac{x_1 x_3 - x_2^2}{2}\frac{\partial}{\partial x_3}.$$

Siegel upper half-space:

$$\mathbb{H}_g = \{\tau \in M_{g\times g}(\mathbb{C}) \mid \tau = \tau^t,\ \Im\tau > 0\}$$

We construct a holomorphic map with "Fourier coefficients in $\mathbb{Z}$" :

$$\varphi_g : \mathbb{H}_g \longrightarrow \mathcal{B}_g(\mathbb{C})$$

satisfying the higher Ramanujan equations:

$$\frac{1}{2\pi i}\frac{\partial\varphi_g}{\partial\tau_{kl}} = v_{kl} \circ \varphi_g .$$

Example (g=1)

Under the previous identification, $\varphi_1 = (E_2, E_4, E_6)$.

Theorem

Let (A, λ) be defined over $\mathbb{Q}$. Then there exists $\tau \in \mathbb{H}_g$ such that

$$\overline{\mathbb{Q}}(Periods(A)) \supset \overline{\mathbb{Q}}(2\pi i, \tau, \varphi_g(\tau))$$

is a finite field extension.

Question: can we extend Nesterenko's methods to φ_2?

Would prove algebraic independence of $\pi, \Gamma(1/5), \Gamma(2/5)$... Note:

generically,

$$G_{mot}(A) = \mathrm{GSp}_{2g} \Longrightarrow \dim G_{mot}(A) = 2g^2 + g + 1$$

By the period conjecture, we expect $\varphi_g(\mathbb{H}_g)$ to be Zariski-dense in $\mathcal{B}_g(\mathbb{C})$.

Theorem

Every analytic leaf of $\mathcal{R}_g$ is Zariski-dense in $\mathcal{B}_g(\mathbb{C})$.

Related to Nesterenko's "D-property" in transcendence theory.

Proof of the special case $\varphi_g(\mathbb{H}_g)$.

- It suffices: $\varphi_g(\mathbb{H}_g)$ is Zariski-dense in each fiber of $\pi : \mathcal{B}_g \to \mathcal{A}_g$. Note: $\mathcal{A}_g(\mathbb{C}) = \mathrm{Sp}_{2g}(\mathbb{Z})\backslash\backslash\mathbb{H}_g$.
- Given $\tau \in \mathbb{H}_g$, boils down to the Zariski-density of

$$\left\{ \begin{pmatrix} (C\tau + D)^{-1} & -\frac{1}{2\pi i} C^t \\ 0 & (C\tau + D)^t \end{pmatrix} \in P_g(\mathbb{C}) \; ; \; \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbb{Z}) \right\}$$

  in $P_g(\mathbb{C})$.
- Follows from the Zariski-density of $\mathrm{Sp}_{2g}(\mathbb{Z})$ in $\mathrm{Sp}_{2g}(\mathbb{C})$.

□

Theorem

The graph of φ_g

$$\{(\tau, \varphi_g(\tau)) \in Sym_g(\mathbb{C}) \times \mathcal{B}_g(\mathbb{C}) \mid \tau \in \mathbb{H}_g\}$$

is Zariski-dense in $Sym_g(\mathbb{C}) \times \mathcal{B}_g(\mathbb{C})$.

([10]) In fact, J.-P. Serre pointed out to me that for an algebraic curve over **C**, these " periods of differentials of the second kind " are rather classical invariants. Thus, for an elliptic curve defined by the periods ω_1, ω_2 one defines classically the integrals

$$\eta_i = \int_0^{\omega_i} \eta,$$

(where $x = \wp z$, $y = \wp' z$, and $\eta = \frac{xdx}{y}$ is a differential of the second kind which, together with the invariant differential ω, forms a basis of $H^1(X)$ = differentials of second kind mod. exact differentials). The only known general algebraic relation among the η_i and ω_i is

$$\omega_1\eta_2 - \eta_1\omega_2 = 2i\pi.$$

Schneider's theorem states that if X is algebraic (i.e. its coefficients g_2 and g_3 are algebraic), then ω_1 and ω_2 are transcendental, and it is believed that if X has no complex multiplication, then ω_1 and ω_2 are algebraically independent. This conjecture extends in an obvious way to the set of periods $(\omega_1, \omega_2, \eta_1, \eta_2)$ and can be rephrased also for curves of any genus, or rather for abelian varieties of dimension g, involving $4g$ periods.

The only general algebraic relation between periods of principally polarized abelian varieties are the ones given by the polarization data.