Calculus on Schemes - Lecture 1

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Contents

1 Introduction

The main objectives of this course are:

1. To introduce some differential techniques in algebraic geometry.
2. To show how to associate to any family of algebraic varieties a certain differential equation.

For instance, let us consider the *Legendre family* of elliptic curves:

$$\begin{array}{c} E \\ \big\downarrow \\ \mathbf{P}^1 \setminus \{0,1,\infty\} \end{array}$$

where the fiber of a point $\lambda \in \mathbf{P}^1 \setminus \{0,1,\infty\}$ is given by the elliptic curve

$$E_\lambda : y^2 = x(x-1)(x-\lambda).$$

To this family is associated a *Picard-Fuchs* equation; namely, a second order differential equation given explicitly by

$$\lambda(1-\lambda)\frac{d}{d\lambda^2} + (1-2\lambda)\frac{d}{d\lambda} - \frac{1}{4} = 0.$$

The solutions of Picard-Fuchs equations are given by *period integrals*, and these encode a lot of information about the given family of algebraic varieties. In the above example, we have the holomorphic solution

$$\varpi(\lambda) = 2\int_1^\infty \frac{dx}{\sqrt{x(x-1)(x-\lambda)}} = 2\pi \sum_{n=0}^{\infty} \binom{-1/2}{n}^2 \lambda^n,$$

and Igusa famously remarked that, for any $\lambda \in \mathbf{Z} \setminus \{0,1\}$ and prime $p > 2$,

$$|\overline{E}_\lambda(\mathbf{F}_p)| = (-1)^{\frac{p+1}{2}} \sum_{n=0}^{\frac{p-1}{2}} \binom{-1/2}{n}^2 \lambda^n \mod p.$$

That is, the number of $\mathbf{F}_p$-points of the fibers are given by a certain truncation of the holomorphic solution ϖ of the Picard-Fuchs equation! This is strong indication that this equation is of 'motivic origin', in some sense.

The above phenomenon was one of the main inspirations for Dwork's work on zeta functions of algebraic varieties via p-adic analysis, and it was greatly clarified by Manin with the introduction of (what we call nowadays) the *Gauss-Manin connection*: a purely algebraic operation allowing to differentiate cohomology classes.

The Gauss-Manin connection is now pervasive in algebraic geometry and in number theory. It is related to the study of periods, modular forms, and mirror symmetry, to name a few. This course is an introduction to this concept.

2 Derivations

The usual derivative

$$f(t) \longmapsto f'(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

of a differentiable real function is $\mathbf{R}$-linear and satisfies the so-called *product rule*:

$$(fg)' = fg' + f'g.$$

Since the above kind of limits are not available in commutative algebra, the main idea in developping an algebraic version of differential calculus is to take the above two (algebraic) properties as an abstract definition of a "derivation".

Definition 2.1. Let R be a ring, A be an R-algebra, and M be an A-module. An R-linear map $D : A \longrightarrow M$ is an *R-derivation* if for every $f, g \in A$ we have

$$D(fg) = fD(g) + gD(f).$$

In particular, $D(r) = 0$ for every $r \in R$; we say that r is a *constant* for D.

The set of such maps is denoted by $\mathrm{Der}_R(A, M)$ and has a natural structure of a left A-module. If $M = A$, then we denote simply $\mathrm{Der}_R(A)$; this is the *A-module of R-derivations of A*.

Example 2.2. If $A = R[x_1, \ldots, x_n]$, then the formal derivative with respect to the 'variable' x_i is the unique R-derivation $\frac{\partial}{\partial x_i}$ of A satisfying

$$\frac{\partial}{\partial x_i}(x_j) = \delta_{ij}.$$

In general, for $f \in A$, we denote

$$\frac{\partial}{\partial x_i}(f) = \frac{\partial f}{\partial x_i}.$$

For any $D \in \mathrm{Der}_R(A)$, we have

$$D = \sum_{i=1}^{n} D(x_i) \frac{\partial}{\partial x_i}.$$

In particular, the A-module $\mathrm{Der}_R(A)$ is free with basis $(\partial/\partial x_i)_{i=1,\dots,n}$:

$$\mathrm{Der}_R(A) = A\frac{\partial}{\partial x_1} \oplus \cdots \oplus A\frac{\partial}{\partial x_n}.$$

When $n = 1$, i.e., $A = R[x]$, and $f = \sum_i r_i x^i \in A$, we also denote

$$f' = \frac{df}{dx} = \frac{\partial f}{\partial x} = \sum_i i r_i x^{i-1}.$$

Remark 2.3 (Geometric interpretation)**.** If $X = \operatorname{Spec} A$, then a derivation $D \in \mathrm{Der}_R(A)$ can also be called a *vector field* on the R-scheme X. For instance, take $R = \mathbf{C}$, and $A = \mathbf{C}[x, y]$. Then R-derivations of A correspond to what is classically known as 'polynomial vector fields on $\mathbf{C}^2$'; namely, expressions of the form

$$v = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y}$$

with $P, Q \in \mathbf{C}[x, y]$. For each $p = (x_0, y_0) \in \mathbf{C}^2$,

$$v_p = P(x_0, y_0) \left.\frac{\partial}{\partial x}\right|_p + Q(x_0, y_0) \left.\frac{\partial}{\partial y}\right|_p$$

should be visualized as a 'tangent vector' of $\mathbf{C}^2$ at p with coordinates $(P(x_0, y_0), Q(x_0, y_0))$.

In general, it is not easy to compute the module of derivations on a given R-algebra. To remedy this, we deal with the better behaved dual notion of differentials.

3 Differentials

Lemma 3.1. *Let R be a ring and A be an R-algebra. The endofunctor on the category of A-modules*

$$M \longmapsto \mathrm{Der}_R(A, M)$$

is representable. That is, there exists a unique (up to unique isomorphism) A-module $\Omega^1_{A/R}$ such that, for every A-module M,

$$\mathrm{Der}_R(A, M) \cong \mathrm{Hom}_A(\Omega^1_{A/R}, M)$$

functorially in M.

Proof. Let $\Omega^1_{A/R}$ be the quotient of the free A-module generated by the symbols df, $f \in A$, by the relations

1. $d(f + g) = df + dg$, for every $f, g \in A$
2. $d(fg) = f dg + g df$, for every $f, g \in A$
3. $dr = 0$, for every $r \in R$.

Then $d : A \longrightarrow \Omega^1_{A/R}$ defines an R-derivation and, for every A-module M with an R-derivation $D : A \longrightarrow M$, there exists a unique A-linear map $\Omega^1_{A/R} \longrightarrow M$ making the following diagram commute:

$$\begin{array}{ccc} \Omega^1_{A/R} & \longrightarrow & M \\ {\scriptstyle d}\uparrow & \nearrow_{D} & \\ A & & \end{array}$$

■

Note that $d : A \longrightarrow \Omega^1_{A/R}$ in the above proof is the 'universal R-derivation' corresponding to $\mathrm{id} \in \mathrm{Hom}_A(\Omega^1_{A/R}, \Omega^1_{A/R})$.

Definition 3.2. The A-module $\Omega^1_{A/R}$ of the above lemma is the module of *differential 1-forms* of A over R.

Example 3.3. If $A = R[x_1, \dots, x_n]$, then $\Omega^1_{A/R}$ is the free module on $(dx_i)_{i=1,\dots,n}$:

$$\Omega^1_{A/R} = Adx_1 \oplus \cdots \oplus Adx_n.$$

The derivation $d : A \longrightarrow \Omega^1_{A/R}$ is given by

$$df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n.$$

In general, the universal property of $\Omega^1_{A/R}$ implies that $\mathrm{Der}_R(A)$ is its dual A-module:

$$(\Omega^1_{A/R})^\vee = \mathrm{Hom}_A(\Omega^1_{A/R}, A) \cong \mathrm{Der}_R(A),$$

the duality being given by the unique A-bilinear pairing

$$\langle \ , \ \rangle : \mathrm{Der}_R(A) \times \Omega^1_{A/R} \longrightarrow A$$

satisfying

$$\langle D, df \rangle = Df$$

for every $f \in A$.

Example 3.4. By Example 3.3, the basis $(\partial/\partial x_i)_{i=1,\dots,n}$ of $\mathrm{Der}_R(R[x_1, \dots, x_n])$ is simply the dual basis of $(dx_i)_{i=1,\dots,n}$.

Let $\varphi : A \longrightarrow B$ be a morphism of R-algebras. Then we have canonical B-linear maps

$$\begin{aligned} \Omega^1_{A/R} \otimes_R B &\longrightarrow \Omega^1_{B/R} \\ df \otimes g &\longmapsto g d\varphi(f) \end{aligned}$$

and

$$\begin{aligned} \Omega^1_{B/R} &\longrightarrow \Omega^1_{B/A} \\ dg &\longmapsto dg. \end{aligned}$$

Proposition 3.5 (Basic exact sequences)**.** *Let $\varphi : A \longrightarrow B$ be a morphism of R-algebras.*

1. *The sequence of B-modules*

$$\Omega^1_{A/R} \otimes_A B \longrightarrow \Omega^1_{B/R} \longrightarrow \Omega^1_{B/A} \longrightarrow 0$$

is exact.

2. *If φ is surjective, so that $B = A/I$ for some ideal $I \subset A$, then $\Omega^1_{B/A} = 0$ and the following sequence of B-modules is exact:*

$$\begin{aligned} I/I^2 &\longrightarrow \Omega^1_{A/R} \otimes_A B \longrightarrow \Omega^1_{B/R} \longrightarrow 0 \\ f + I^2 &\longmapsto df \otimes 1 \end{aligned}$$

Proof. The exactness of the first exact sequence is equivalent to the exactness of

$$0 \longrightarrow \mathrm{Hom}_B(\Omega^1_{B/A}, N) \longrightarrow \mathrm{Hom}_B(\Omega^1_{B/R}, N) \longrightarrow \mathrm{Hom}_B(\Omega^1_{A/R} \otimes_R B, N)$$

for every B-module N. By the universal property of differential forms, the above sequence is isomorphic to

$$0 \longrightarrow \mathrm{Der}_A(B, N) \longrightarrow \mathrm{Der}_R(B, N) \longrightarrow \mathrm{Der}_R(A, N),$$

which is easily checked to be exact.

If $B = A/I$, then $\Omega^1_{B/A} = 0$ since B is generated by constants. Here again, we use the universal property and we check that

$$\begin{aligned} 0 \longrightarrow \mathrm{Der}_R(A/I, N) \longrightarrow \mathrm{Der}_R(A, N) &\longrightarrow \mathrm{Hom}_{A/I}(I/I^2, N) \\ D &\longmapsto (f + I^2 \longmapsto D(f)) \end{aligned}$$

is exact for every A/I-module N. ■

Example 3.6 (Affine hypersurfaces)**.** Let $A = R[x_1, \ldots, x_n]$, $f \in A$, and $B = A/(f)$. Then

$$\Omega^1_{B/R} = \frac{Bdx_1 \oplus \cdots \oplus Bdx_n}{Bdf}.$$

Example 3.7 (Standard étale algebras)**.** Let $f, g \in R[x]$ with f monic, and set

$$A = R[x]_g/(f).$$

If the image of f' in A is invertible, then we say that A is a *standard étale algebra* over R. In this case it follows from the last example that $\Omega^1_{A/R} = 0$.

Remark 3.8. In the above example, $X = \mathrm{Spec}\, A$ should be seen as a "covering space" of $S = \mathrm{Spec}\, R$. For instance, take $R = \mathbf{C}[t, t^{-1}]$, $f = x^n - t$ (for some $n \geq 1$), and $g = x$. Then $R \longrightarrow A$ is standard étale and the morphism $X \longrightarrow S$ induces on the level of $\mathbf{C}$-points the usual n-sheeted covering

$$\mathbf{C}^\times \longrightarrow \mathbf{C}^\times, \qquad z \longmapsto z^n.$$

We now provide an alternative way of defining $\Omega^1_{A/R}$.

Proposition 3.9 (Differentials via the augmentation ideal)**.** *Let A be an R-algebra and $I \subset A \otimes_R A$ (the* augmentation ideal*) be the kernel of the R-morphism*

$$A \otimes_R A \longrightarrow A, \qquad f \otimes g \longmapsto fg$$

Then:

1. *The left and right A-algebra structures on $A \otimes_R A$ induce the same A-module structure on I/I^2;*
2. *The map*
$$\delta : A \longrightarrow I/I^2, \qquad f \longmapsto 1 \otimes f - f \otimes 1 \mod I^2$$
is an R-derivation and its image generates I/I^2 as an A-module.
3. *The A-linear map $\Omega^1_{A/R} \longrightarrow I/I^2$ induced by δ is an isomorphism.*

In other words, I/I^2 can be identified with the module of differential 1-forms $\Omega^1_{A/R}$, with universal derivation $d : A \longrightarrow \Omega^1_{A/R}$ being given by $f \longmapsto 1\otimes f + f \otimes 1 \mod I^2$.

Proof. For any $f \in A$, $1 \otimes f - f \otimes 1 \in I$. Thus, for $\alpha \in I$, we have

$$(1 \otimes f)\alpha - \alpha(f \otimes 1) = (1 \otimes f - f \otimes 1)\alpha \in I^2$$

This proves 1.

Using 1, it is easy to check that δ is an R-derivation. The rest of 2 follows from the following formula:

$$\sum_i f_i \otimes g_i = \sum_i (f_i \otimes 1)(1 \otimes g_i - g_i \otimes 1) + \sum_i f_i g_i \otimes 1.$$

To prove 3, consider the A-module $P = A \oplus \Omega^1_{A/R}$ and define a ring structure on P by

$$(f, \omega) \cdot (g, \eta) = (fg, f\eta + g\omega).$$

The ring P is an A-algebra via $f \longmapsto (f, 0)$. Note that $\Omega^1_{A/R}$ can be naturally identified with an ideal $J = \{(0, \omega) \in P \mid \omega \in \Omega^1_{A/R}\}$ of P satisfying $J^2 = 0$. Clearly,

$$A \otimes_R A \longrightarrow P, \qquad f \otimes g \longmapsto (fg, fdg)$$

is a well-defined (left) A-algebra morphism sending I to J, so that it induces an A-linear map

$$I/I^2 \longrightarrow \Omega^1_{A/R}$$

sending $\delta(f)$ to df, that is, an inverse of $\Omega^1_{A/R} \longrightarrow I/I^2$. ■

4 Globalizing

Definition 4.1. Let X be an S-scheme and denote by $\mathcal{I}$ the ideal sheaf defined by the diagonal immersion[1] $X \longrightarrow X \times_S X$. The sheaf of differential 1-forms of X over S is the quasi-coherent $\mathcal{O}_X$-module defined by

$$\Omega^1_{X/S} = \mathcal{I}/\mathcal{I}^2.$$

If $p_j : X \times_S X \longrightarrow X$ denotes the jth projection, $j = 1, 2$, then we define an $\mathcal{O}_S$-morphism

$$d : \mathcal{O}_X \longrightarrow \Omega^1_{X/S}, \qquad f \longmapsto p_2^* f - p_1^* f \mod \mathcal{I}^2.$$

It follows from Proposition 3.9 that for any affine open $U = \operatorname{Spec} A \subset X$ lying over an affine open $V = \operatorname{Spec} R \subset S$, we have

$$\Gamma(U, \Omega^1_{X/S}) = \Omega^1_{A/R},$$

and that d restricts to the universal derivation $A \longrightarrow \Omega^1_{A/R}$. Similarly, for every $x \in X$ lying over $s \in S$,

$$(\Omega^1_{X/S})_x = \Omega^1_{\mathcal{O}_{X,x}/\mathcal{O}_{S,s}}.$$

Exercise 4.2 (Vector bundles)**.** Let $p : \mathbb{V}(\mathcal{E}) \longrightarrow S$ be the total space of a vector bundle[2] $\mathcal{E}$ over a scheme S. Prove that

$$\Omega^1_{E/S} \cong p^*\mathcal{E}$$

canonically. Hint: prove first that for any R-modules M and N, we can identify $\operatorname{Hom}_R(M, N) = \operatorname{Der}_R(\operatorname{Sym} M, N)$ and conclude that $\operatorname{Sym} M \otimes_R M \longrightarrow \Omega^1_{\operatorname{Sym} M/R}$, $f \otimes m \longmapsto fdm$, is an isomorphism of $\operatorname{Sym} M$-modules.

[1] In general, an *immersion* $i : Z \hookrightarrow X$ is an isomorphism over a closed subscheme $i(Z)$ of a largest open subset U of X; the ideal $\mathcal{I}$ of i is by definition the quasi-coherent sheaf of ideals defining $i(Z)$ in U.

[2] Here, a *vector bundle* means a locally free sheaf of finite rank.

Example 4.3 (Punctured elliptic curve)**.** Let k be a field of characteristic $\neq 2, 3$, and U be the closed affine subscheme of $\mathbf{A}_k^2 = \operatorname{Spec} k[x, y]$ defned by the equation

$$f = y^2 - (4x^3 - g_2 x - g_3) \in k[x, y]$$

with $g_2, g_3 \in k$ satisfying $g_2^3 - 27g_3^2 \neq 0$. This last condition implies that

$$df = 2ydy - (12x^2 - g_2)dx \in \Omega^1_{\mathbf{A}_k^2/k}$$

never vanishes on U. In particular, $U_0 = D(y) \cap U$ and $U_1 = D(12x^2 - g_2) \cap U$ define an open covering of X. Since $df = 0$ in $\Omega^1_{U/k}$ (cf. Example 3.6), there's a unique global section $\omega \in \Gamma(U, \Omega^1_{U/k})$ such that

$$\omega|_{U_0} = \frac{dx}{y} \qquad \omega|_{U_1} = \frac{2dy}{12x^2 - g_2}.$$

Actually, it follows from Example 3.6 that $\Omega^1_{U/k}|_{U_0}$ is free of rank 1 with generator $\omega|_{U_0}$, and similarly for U_1. This proves that $\Omega^1_{U/k}$ is line bundle over U trivialized by ω. By abuse, it is common to simply denote

$$\omega = \frac{dx}{y}.$$

Exercise 4.4. Generalize the above example to any affine plane curve.

Let us now consider some projective examples.

Example 4.5 (Projective line)**.** Consider $\mathbf{P}_R^1 = \operatorname{Proj} R[x_0, x_1]$, with open covering $U_0 = D_+(x_0) = \operatorname{Spec} R[x_1/x_0]$ and $U_1 = D_+(x_1) = \operatorname{Spec} R[x_0/x_1]$. Then there exists a unique global section $\omega \in \Gamma(\mathbf{P}_R^1, \mathcal{O}(2) \otimes \Omega^1_{\mathbf{P}_R^1/R})$ such that

$$\omega|_{U_0} = x_0^2 d\left(\frac{x_1}{x_0}\right) \qquad \omega|_{U_1} = -x_1^2 d(\frac{x_0}{x_1}).$$

Indeed, one may readily verify that the above forms coincide on $U_0 \cap U_1$. Since x_i^2 trivializes $\mathcal{O}(2)|_{U_i}$ and $d(x_{1-i}/x_i)$ trivializes $\Omega^1_{U_i/R}$ (see Example 3.3), ω is a trivialization of $\mathcal{O}(2) \otimes \Omega^1_{\mathbf{P}_R^1/R}$. In particular,

$$\Omega^1_{\mathbf{P}_R^1/R} \cong \mathcal{O}(-2).$$

In particular, $\Omega^1_{\mathbf{P}_R^1/R}$ admits no global section.

Example 4.6 (Elliptic curve)**.** Let k be a field of characteristic $\neq 2, 3$ and $E \subset \mathbf{P}_k^2 = \operatorname{Proj} k[X, Y, Z]$ be the projective plane curve defined by the homogeneous equation of degree 3

$$F = Y^2 Z - (4X^3 - g_2 X Z^2 - g_3 Z^3) \in k[X, Y, Z].$$

with $g_2^3 - 27g_3^2 \neq 0$. We denote $U = E \cap D_+(Z)$ and $(X/Z, Y/Z) = (x, y)$, so that U coincides with the affine curve considered in Example 4.3. We will prove that the differential form

$$\omega = \frac{dx}{y}$$

on U extends to a global differential form on E, and that it trivializes $\Omega^1_{E/k}$; that is,

$$\Omega^1_{E/k} = \mathcal{O}_E \frac{dx}{y}.$$

Indeed, let $V = E \cap D_+(Y)$, so that $E = U \cup V$. If we denote $(X/Y, Z/Y) = (t, s)$, then V is the affine curve given by the equation

$$s - (4t^3 - g_2 ts - g_3 s^3) \in k[t, s].$$

Arguing as before, we see that

$$-\frac{2ds}{12t^2 - g_2 s^2}$$

trivializes $\Omega^1_{V/k}$. Now, over $U \cap V$, we have $(t, s) = (x/y, 1/y)$, so that

$$-\frac{2ds}{12t^2 - g_2 s^2} = -\frac{2d(1/y)}{12(x/y)^2 - g_2(1/y)^2} = \frac{2dy}{12x^2 - g_2} = \omega.$$

Thus, ω extends to a global section of $\Omega^1_{E/k}$. It trivializes this line bundle globally since it does locally.