Calculus on Schemes - Lecture 2

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Contents

1 Globalizing (cont.)

We come back to the general theory. Let S be a base scheme. There are two main situations in which we want to study differetials:

(a) A morphism of S-schemes:

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

Here, X should be thought as a family of S-schemes $(X_y)_{y \in Y}$ parametrized by Y.

(b) An S-immersion of ideal $\mathcal{I}$:

$$\begin{array}{ccc} Z & \xhookrightarrow{i} & X \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

That is, Z is an S-subscheme of the "ambient S-scheme" X.

Proposition 1.1. *In situation (a), we have a canonical exact sequence*

$$\varphi^* \Omega^1_{Y/S} \longrightarrow \Omega^1_{X/S} \longrightarrow \Omega^1_{X/Y} \longrightarrow 0.$$

In situation (b), we denote $C_{Z/X} = \mathcal{I}/\mathcal{I}^2$, and we have a canonical exact sequence

$$C_{Z/X} \xrightarrow{d} i^* \Omega^1_{X/S} \longrightarrow \Omega^1_{Z/S} \longrightarrow 0.$$

The sheaf $C_{Z/X}$ is called the conormal sheaf of Z in X.

Proof. It follows immediately from the corresponding affine statement in the last lecture (Basic exact sequences). ■

Finally, we state the compatibility of the sheaf of differentials with products and pullbacks.

Proposition 1.2. *1. Consider a Cartesian square of schemes*

$$\begin{array}{ccc} X' & \xrightarrow{\varphi} & X \\ \downarrow & \square & \downarrow \\ S' & \longrightarrow & S \end{array}$$

Then we have a canonical isomorphism

$$\Omega^1_{X'/S'} \cong \varphi^*\Omega^1_{X/S}.$$

2. Let X_1 and X_2 be S-schemes, then

$$\Omega^1_{X_1\times_S X_2/S} = \Omega^1_{X_1\times_S X_2/X_2} \oplus \Omega^1_{X_1\times_S X_2/X_1}.$$

In particular, if $p_i : X_1 \times_S X_2 \longrightarrow X_i$ denotes the ith projection, $i = 1, 2$, then it follows from 1 that we have a canonical isomorphism

$$\Omega^1_{X_1\times_S X_2/S} \cong p_1^*\Omega^1_{X_1/S} \oplus p_2^*\Omega^1_{X_2/S}.$$

Proof. Exercise. ■

We next turn our attention to a class of morphisms which allow us to get more precise information on differentials.

2 Smooth morphisms

Definition 2.1. We say that a closed immersion $X \hookrightarrow X_1$ is a *thickening* (of order 1) of X if its defining ideal $\mathcal{I} \subset \mathcal{O}_X$ satisfies $\mathcal{I}^2 = 0$. We say that the thickening $X \hookrightarrow X_1$ is *affine* if both X and X_1 are affine schemes.

Example 2.2 (Dual numbers)**.** Let R be a ring, $X = \operatorname{Spec} R$, and $X_1 = \operatorname{Spec} R[\epsilon]$ where $\epsilon^2 = 0$ (i.e., $R[\epsilon] = R[x]/(x^2)$). Then the closed immersion $X \hookrightarrow X_1$ induced by the R-algebra morphism

$$\begin{aligned} R[\epsilon] &\longrightarrow R \\ \epsilon &\longmapsto 0 \end{aligned}$$

is an affine thickening. The ring $R[\epsilon]$ is called 'ring of dual numbers over R'.

Definition 2.3 (Smoothness)**.** Let $\pi : X \longrightarrow S$ be a morphism of schemes. We say that π is *smooth* if

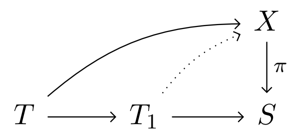
1. π is locally of finite presentation[1];

2. for every diagram

$$\begin{array}{ccccc} & & & \nearrow & X \\ & & & & \downarrow \pi \\ T & \longrightarrow & T_1 & \longrightarrow & S \end{array}$$

[1] This means that for every $p \in X$, there exists an affine open neighborhood $U = \operatorname{Spec} A$ of x and an affine open neighborhood $V = \operatorname{Spec} R$ of $\pi(x)$ such that $\pi^* : R \longrightarrow A$ is of finite presentation. If S is locally Noetherian, then "locally of finite presentation" is equivalent to "locally of finite type".

where $T \hookrightarrow T_1$ is an affine thickening, there exists a morphism $T_1 \longrightarrow X$ making



commute.

When π is implicit, we also say that X is smooth over S, or that the S-scheme X is smooth.

Remark 2.4. Smoothness is equivalent to $\mathrm{Hom}_S(T_1, X) \longrightarrow \mathrm{Hom}_S(T, X)$ being surjective for any affine S-thickening $T \hookrightarrow T_1$.

Smoothness is a local property on X.[2] If $p \in X$, we say that π is smooth at p if there exists a open neighborhood U of p such that $\pi|_U : U \longrightarrow S$ is smooth.

It also follows immediately from the definition that *smoothness is preserved by base change*: for any Cartesian diagram of schemes

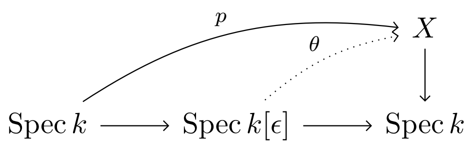
$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & \square & \downarrow \\ S' & \longrightarrow & S \end{array}$$

if $X \longrightarrow S$ is smooth, then $X' \longrightarrow S'$ is smooth.

Example 2.5. To start to get a feeling of what's going on, consider an affine plane curve over a field k defined by some $f \in k[x, y]$:

$$X = \mathrm{Spec}\, k[x, y]/(f)$$

Let us consider a diagram



where $\mathrm{Spec}\, k \longrightarrow \mathrm{Spec}\, k[\epsilon]$ is the affine thickening defined in Example 2.2. Note that $p = (a, b)$ corresponds to a point in k^2 such that $f(p) = f(a, b) = 0$. A lifting θ of p corresponds to a k-morphism

$$\theta^* : k[x, y]/(f) \longrightarrow k[\epsilon]$$

such that $\theta^*(x) = a + u\epsilon$, $v^*(y) = b + v\epsilon$, with $(u, v) \in k^2$ satisfying

$$f(a + u\epsilon, b + v\epsilon) = 0.$$

Since $\epsilon^2 = 0$, we conclude from the Taylor formula

$$f(x, y) = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b) + g, \qquad g \in ((x - a)^2, (x - a)(y - b), (y - b)^2)$$

that

$$f(a + u\epsilon, b + v\epsilon) = \left(\frac{\partial f}{\partial x}(a, b)u + \frac{\partial f}{\partial y}(a, b)v\right)\epsilon$$

[2]Update: actually, this is not immediate from this definition (see exercise sheet).

Thus, liftings θ of $p = (a, b)$ correspond to pairs $(u, v) \in k^2$ such that

$$\frac{\partial f}{\partial x}(a,b)u + \frac{\partial f}{\partial y}(a,b)v = 0.$$

These are, intuitivelly, tangent vectors of X at p.

Remark 2.6. Incidentally, if morphisms $\operatorname{Spec} k[\epsilon] \longrightarrow S$ are to be seen as tangent vectors of S, then smoothness of a morphism $X \longrightarrow S$ implies that every tangent vector of S can be lifted to X. This is how a "submersion" is defined in differential geometry. We will make this more precise in the next lecture.

In the next example we characterize smoothness of an affine plane curve.

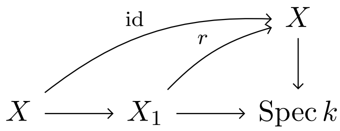
Example 2.7. Let us keep the same notation of last example and let us prove the following: $X = \operatorname{Spec} k[x, y]/(f)$ is smooth over $\operatorname{Spec} k$ if and only if

$$\left(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (1) \tag{2.1}$$

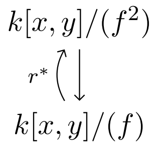
as an ideal of $k[x, y]$. Let us first assume that X is smooth over $\operatorname{Spec} k$. Define

$$X_1 = \operatorname{Spec} k[x, y]/(f^2)$$

so that $X \hookrightarrow X_1$ is an affine thickening. By smoothness, there exists a morphism $r : X_1 \longrightarrow X$ making the diagram



commute; that is, r is a *retraction*. On the level of rings, we obtain a section



Thus

$$r^*(x + (f)) = x + g_1 f + (f^2), \qquad r^*(y + (f)) = y + g_2 f + (f^2)$$

where $g_1, g_2 \in k[x, y]$ satisfy

$$f(x + g_1 f, y + g_2 f) \in (f^2).$$

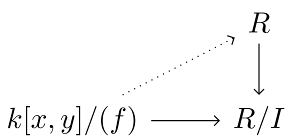
It follows from Taylor's formula that

$$f(x + g_1 f, y + g_2 f) - \left(f + \frac{\partial f}{\partial x} g_1 f + \frac{\partial f}{\partial y} g_2 f\right) \in (f^2)$$

so that

$$1 + \frac{\partial f}{\partial x} g_1 + \frac{\partial f}{\partial y} g_2 \in (f).$$

in $k[x, y]$. This proves that f satisfies (2.1). Conversely, if f, $\partial f/\partial x$, and $\partial f/\partial y$ are coprime, then a section $r^* : T \longrightarrow X$ as above exists. Thus, for any solid diagram of k-algebras



where R is a k-algebra and $I \subset R$ is an ideal such that I^2, we can define a dotted arrow by lifting $k[x,y]/(f) \longrightarrow R/I$ to $k[x,y]/(f^2) \longrightarrow R$ (which is always possible) and composing with r^*:

$$\begin{array}{ccc} k[x,y]/(f^2) & \dashrightarrow & R \\ {\scriptstyle r^*} \uparrow\!\downarrow & & \downarrow \\ k[x,y]/(f) & \longrightarrow & R/I \end{array}$$

This proves that X is smooth over $\operatorname{Spec} k$.

Exercise 2.8 (Jacobian criterion for affine hypersurfaces)**.** Generalize the last example as follows. Prove that if k is a field and $f \in k[x_1, \dots, x_n]$, then the k-scheme $X = \operatorname{Spec} k[x_1, \dots, x_n]/(f)$ is smooth if and only if

$$\left(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right) = (1)$$

as ideals in $k[x_1, \dots, x_n]$.

Example 2.9. $\mathbf{A}^n_S$ is clearly smooth over S. As smoothness is a local property, $\mathbf{P}^n_S$ is also smooth over S.

3 Smoothness and differentials

We start by clarifying the relation between retractions of thickenings and derivations.

Example 3.1 (Retractions and derivations)**.** Consider an affine R-thickening $i : X \hookrightarrow X_1$ of ideal I. If we denote $X = \operatorname{Spec} A$ and $X_1 = \operatorname{Spec} A_1$ (so that $A = A_1/I$), then any pair of retractions $r_1, r_2 : X_1 \longrightarrow X$ of i define a derivation

$$D := r_2^* - r_1^* \in \operatorname{Der}_R(A, I).$$

Indeed, note first that since r_1 and r_2 are both retractions of i, for any $f \in A$, $r_2^*(f)$ and $r_1^*(f)$ map to the same element f in the quotient $A_1/I = A$, so that $D(f) = r_2^*(f) - r_1^*(f) \in I$. Now observe that, since $I^2 = 0$, the A-module structure of I is given by $f \cdot x = f_1 x$, where f_1 is any element of A_1 lifting f (here, $f \in A$ and $x \in I$); in particular, both r_1^* and r_2^* induce the *same* A-module structure on I. It is now easy to check that D is a derivation:

$$\begin{aligned} D(fg) &= r_2^*(f) r_2^*(g) - r_1^*(f) r_1^*(g) \\ &= r_2^*(f)(r_2^*(g) - r_1^*(g)) + r_1^*(g)(r_2^*(f) - r_1^*(f)) \\ &= f D(g) - g D(f). \end{aligned}$$

Remark 3.2. For $X = \operatorname{Spec} A$ over $S = \operatorname{Spec} R$, the definition of $\Omega^1_{A/R}$ via the augmentation ideal shows that the universal derivation $d : A \longrightarrow \Omega^1_{A/R}$ is given by the above procedure. The first infinitesimal neighborhood of the diagonal $\Delta : X \longrightarrow X \times_S X$ gives a thickening of X and d corresponds to the retractions given by the two projections.

Here's the particularly useful special case of thickenings that already come with a retraction.

Example 3.3 (Linear thickenings)**.** Let A be a ring and $X = \operatorname{Spec} A$. For any A-module M, we can define a ring structure on

$$A[M] := A \oplus M$$

by

$$(f, m) \cdot (f', m') = (ff', fm' + f'm).$$

Then M gets identified via $m \longmapsto (0, m)$ to an ideal $A[M]$ such that $M^2 = 0$, and the map $A[M] \longrightarrow A$ given by the first projection induces an isomorphism $A[M]/M \cong A$. Thus

$$X \hookrightarrow \operatorname{Spec} A[M] =: X_1$$

is an affine thickening with ideal M.[3] Note that there is an obvious retraction $X_1 \longrightarrow X$ induced by the inclusion in the first coordinate $A \longrightarrow A[M]$. Thus, by last example, if A is an R-algebra, elements $D \in \operatorname{Der}_R(A, M)$ correspond to retractions $r : X_1 \longrightarrow X$ via $r^*(f) = (f, D(f))$.

Remark 3.4 (Geometric interpretation of linear thickenings)**.** Let X be a scheme, $\mathcal{F}$ be a quasi-coherent sheaf over X and define X_1 to be the 'first infinitesimal neighborhood' of the zero section

$$\begin{array}{c} \mathbf{V}(\mathcal{F}) \\ {\scriptstyle 0}\big\uparrow\big\downarrow \\ X \end{array}$$

Namely, if $\mathcal{I}$ denotes the ideal of the zero section $0 : X \longrightarrow \mathbf{V}(\mathcal{F})$, then X_1 is the subscheme of $\mathbf{V}(\mathcal{F})$ defined by $\mathcal{I}^2$. Then the zero section factors through a closed immersion $X \hookrightarrow X_1$ wich is a thickening of ideal $\mathcal{I}$. Finally, we remark that $\mathcal{I}$ is isomorphic to $\mathcal{F}$ as a quasi-coherent sheaf over X.

We now come to the main theorems of this lecture.

Theorem 3.5. *If X is a smooth S-scheme, then $\Omega^1_{X/S}$ is a vector bundle over X.*

Proof. Since X is of finite presentation over S, the $\mathcal{O}_X$-module $\Omega^1_{X/S}$ is coherent. To prove that $\Omega^1_{X/S}$ is locally free, we can assume that $X = \operatorname{Spec} A$ and $S = \operatorname{Spec} R$ are affine. Thus, we want to prove that $\Omega^1_{A/R}$ is a projective A-module, i.e., that every exact sequence of A-modules

$$0 \longrightarrow M \longrightarrow N \xrightarrow{\alpha} \Omega^1_{A/R} \longrightarrow 0 \tag{3.1}$$

splits.

Given an extension of $\Omega^1_{A/R}$ by M as above, we construct a thickening of $X = \operatorname{Spec} A$ with ideal M as follows (see Example 3.3). Consider the subring

$$A_1 = \{(f, n) \in A[N] \mid df = \alpha(n)\} \leq A[N].$$

Note that $M = A_1 \cap N$ is an ideal of A satisfying $M^2 = 0$, and that $A = A_1/M$ via the first projection $A_1 \longrightarrow A$. Thus the induced morphism

$$i : X = \operatorname{Spec} A \hookrightarrow \operatorname{Spec} A_1 =: X_1$$

is a thickening of X with ideal M.

If X is smooth over S, then there exists $r : X_1 \longrightarrow X$ such that

$$\begin{array}{ccccc} & & & & X \\ & & & {\scriptstyle \text{id}}\nearrow \ \ {\scriptstyle r}\nearrow & \big\downarrow \\ X & \xrightarrow{i} & X_1 & \longrightarrow & S \end{array}$$

[3] In general, an affine thickening $X \hookrightarrow X_1$ with ideal M is also called an *extension of X by M*; the extension given by $A[M]$ as above is called the *trivial extension.*

commutes; that is, $r : X_1 \longrightarrow X$ is a retraction of $i : X \hookrightarrow X_1$. In particular, $r^* : A \longrightarrow A_1$ is an R-algebra morphism of the form

$$f \longmapsto (f, D(f))$$

for some $D \in \mathrm{Der}_R(A, N)$ satisfying $\alpha(D(f)) = df$. By the universal property of $\Omega^1_{A/R}$, D induces a splitting of (3.1). ∎

Definition 3.6. The *relative dimension* of a smooth morphism $\pi : X \longrightarrow S$ is the rank of the vector bundle $\Omega^1_{X/S}$. If $\pi : X \longrightarrow S$ is only smooth at $p \in X$, then the relative dimension of π at p is the rank of $\Omega^1_{X/S,p}$.

We will relate the notion of relative dimension with the usual dimension theory for schemes in the next lecture.

Theorem 3.7. *Consider an S-morphism $\varphi : X \longrightarrow Y$. If φ is smooth, then*

$$0 \longrightarrow \varphi^*\Omega^1_{Y/S} \longrightarrow \Omega^1_{X/S} \longrightarrow \Omega^1_{X/Y} \longrightarrow 0$$

is exact and locally split.

Proof. We can assume everything is affine: $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$, and $S = \operatorname{Spec} R$. It is sufficient to prove that for any B-module N, the sequence

$$0 \longrightarrow \mathrm{Der}_A(B, N) \longrightarrow \mathrm{Der}_R(B, N) \longrightarrow \mathrm{Der}_R(A, N) \longrightarrow 0$$

is split exact, or yet that the restriction map

$$\begin{aligned} \mathrm{Der}_R(B, N) &\longrightarrow \mathrm{Der}_R(A, N) \\ D &\longmapsto D|_A \end{aligned}$$

admits a section.

Let $B_1 = B[N]$. Any $D \in \mathrm{Der}_R(A, N)$ induces an A-algebra structure

$$\begin{aligned} A &\longrightarrow B_1 \\ f &\longmapsto (f, D(f)) \end{aligned}$$

on B_1 making the first projection $B_1 \longrightarrow B$ a morphism of A-algebras. Since $X = \operatorname{Spec} B$ is smooth over $Y = \operatorname{Spec} A$, there exists a retraction $r : X_1 := \operatorname{Spec} B_1 \longrightarrow X$. The morphism $r^* : B \longrightarrow B_1$ is of the form $g \longmapsto (g, \tilde{D}(g))$ for a unique R-derivation $\tilde{D} : B \longrightarrow N$. Finally, we can check that

$$D \longmapsto \tilde{D}$$

defines a section of the restriction $\mathrm{Der}_R(B, N) \longrightarrow \mathrm{Der}_R(A, N)$. ∎

Theorem 3.8 (Conormal exact sequence)**.** *Consider an S-immersion $i : Z \hookrightarrow X$. If Z is smooth over S, then*

$$0 \longrightarrow C_{Z/X} \xrightarrow{d} i^*\Omega^1_{X/S} \longrightarrow \Omega^1_{Z/S} \longrightarrow 0.$$

is exact and locally split.

Proof. We can assume everything is affine: $S = \operatorname{Spec} R$, $X = \operatorname{Spec} A$, and $Z = \operatorname{Spec} A/I$ for some ideal $I \subset A$. Consider the first infinitesimal neighborhood of Z in X, i.e., the subscheme $Z_1 = \operatorname{Spec} A/I^2$ defined by I^2. Thus $i : Z \hookrightarrow X$ factors through a thickening $i_1 : Z \hookrightarrow Z_1$.

Since Z is smooth over S, there exists a retraction $r : Z_1 \longrightarrow Z$ of $i_1 : Z \hookrightarrow Z_1$, which yields $D \in \operatorname{Der}_R(A/I^2, I/I^2)$ defined by

$$D(f) = f - (i \circ r)^*(f).$$

One can now check that D, seen in $\operatorname{Der}_R(A, I/I^2)$, induces a splitting

$$I/I^2 \longrightarrow \Omega^1_{A/R} \otimes_R A/I \longrightarrow \Omega^1_{(A/I)/R} \longrightarrow 0. \qquad (\text{splitting } \Omega^1_{A/R} \otimes_R A/I \dashrightarrow I/I^2 \text{ given by } \langle D, - \rangle)$$

■

Note that the above proof actually give us more: whenever everything is affine, the conormal sequence is split exact, and the splitting is given by some R-derivation $D : A \longrightarrow I/I^2$.

Example 3.9. Taking $X = \mathbf{A}^n_k$, and Z the hypersurface defined by some $f \in A = k[x_1, \ldots, x_n]$. If Z is smooth over k, then it follows from the above theorem that there exists $D \in \operatorname{Der}_k(A, (f)/(f)^2)$ such that $D(f) = f + (f)^2$. Since any D is a linear combination of $\frac{\partial}{\partial x_i}$, we recover one implication of Exercise 2.8.