Calculus on Schemes - Lecture 3

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1 Unramified and étale morphisms

Definition 1.1 (Unramified morphisms)**.** Let $\pi : X \longrightarrow S$ be a morphism of schemes. We say that π is *unramified* if

1. π is locally of finite presentation;
2. for every diagram
$$\begin{tikzcd} & & X \arrow[d, "\pi"] \\ T \arrow[r] \arrow[urr, bend left] & T_1 \arrow[r] & S \end{tikzcd}$$
where $T \hookrightarrow T_1$ is an affine thickening, there exists *at most one* morphism $T_1 \longrightarrow X$ making
$$\begin{tikzcd} & & X \arrow[d, "\pi"] \\ T \arrow[r] \arrow[urr, bend left] & T_1 \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$
commute.

Remark 1.2. Unramifiedness is equivalent to $\mathrm{Hom}_S(T_1, X) \longrightarrow \mathrm{Hom}_S(T, X)$ being injective for any affine S-thickening $T \hookrightarrow T_1$.

For instance, any immersion is unramified. On the other hand, the following exercise deals with the prototypical example of a "ramified covering".

Exercise 1.3. Let $\pi_n : \mathbf{A}^1_{\mathbf{C}} \longrightarrow \mathbf{A}^1_{\mathbf{C}}$ be defined by $\pi_n^*(t) = t^n$. Then π_n is unramified if and only if $n = 1$.

We can characterize unramifiedness purely in terms of differentials as follows.

Theorem 1.4. *A morphism of schemes $X \longrightarrow S$ is unramified if and only if it is locally of finite presentation and $\Omega^1_{X/S} = 0$.*

Proof. We can assume $X = \operatorname{Spec} A$ and $S = \operatorname{Spec} R$. Recall that for any A-module M, elements of $\operatorname{Der}_R(A, M)$ correspond to S-retractions $X_1 = \operatorname{Spec} A[M] \longrightarrow X$. If X is unramified over S, then there can be only one such retraction: namely the one given by the inclusion $A \longrightarrow A[M]$ in the first coordinate. This proves that $0 = \operatorname{Der}_R(A, M) = \operatorname{Hom}_A(\Omega^1_{A/R}, M)$. Thus $\Omega^1_{A/R} = 0$.

Conversely, if Λ is an R-algebra, $I \subset \Lambda$ is an ideal such that $I^2 = 0$ and $\psi_1, \psi_2 : A \longrightarrow \Lambda$ are two R-algebra morphisms which become equal modulo I, then $D = \psi_2 - \psi_1 : A \longrightarrow I$ gives a non-trivial element of $\operatorname{Der}_R(A, I)$ (here I is seen as an A-module via ψ_1 or ψ_2). Since $\Omega^1_{A/R} = 0$, we must have $D = 0$, i.e., $\psi_1 = \psi_2$. ∎

Definition 1.5 (Étale morphism)**.** We say that a morphism of schemes $X \longrightarrow S$ is *étale* if it is smooth and unramified.

Thus, a morphism $X \longrightarrow S$ is étale if it is locally of finite presentation and if for every solid diagram

$$\begin{tikzcd} & & X \arrow[d, "\pi"] \\ T \arrow[r] \arrow[urr, bend left] & T_1 \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

where $T \longrightarrow T_1$ is a thickening, there exists a unique (dotted) morphism $T_1 \longrightarrow X$ making everything commute. Here, $T \longrightarrow T_1$ is not necessarily affine. This is because the unicity in the definition of étale morphisms actually shows that we can glue liftings on affine pieces.

The basic examples of étale morphisms are open immersions and standard étale algebras. Note that a closed immersion is unramified but is (almost) never étale!

Exercise 1.6. Prove directly from the definition that a standard étale algebra A over R is étale. We recall that $A = R[x]_g/(f)$ where $f, g \in R[x]$, with f monic and such that the image of f' in A is a unit. Hint: same spirit of Newton's method or Hensel's lemma.

An étale morphisms is the analog in algebraic geometry of a "local isomorphism" in differential geometry, although it will *not* be in general a local isomorphism in the category of schemes. The following proposition makes this analogy a bit more precise.

If X is a scheme and $p \in X$, we denote by $\hat{\mathcal{O}}_{X,p}$ the completion of the local ring $\mathcal{O}_{X,p}$ with respect to its maximal ideal $\mathfrak{m}_p$.

Proposition 1.7. *Let $\pi : X \longrightarrow S$ be étale at $p \in X$ and assume that $k_{\pi(p)} \xrightarrow{\sim} k_p$. Then the morphism of local rings $\mathcal{O}_{S,\pi(p)} \longrightarrow \mathcal{O}_{X,p}$ induces an isomorphism*

$$\hat{\mathcal{O}}_{S,\pi(p)} \xrightarrow{\sim} \hat{\mathcal{O}}_{X,p}.$$

Proof. Consider the solid diagram

$$\begin{tikzcd} \operatorname{Spec} k_p \arrow[r] \arrow[d] & \operatorname{Spec} \mathcal{O}_{X,p}/\mathfrak{m}_p^2 \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\ \operatorname{Spec} k_{\pi(p)} \arrow[r] \arrow[u, bend left] & \operatorname{Spec} \mathcal{O}_{S,\pi(p)}/\mathfrak{m}_{\pi(p)}^2 \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

where the bended arrow is given by the inverse of $k_{\pi(p)} \xrightarrow{\sim} k_p$. By smoothness, there exists a dotted arrow making the whole diagram commute. The commutativity of the diagram implies that this dotted arrow factors through a section

$$\operatorname{Spec} \mathcal{O}_{S,\pi(p)}/\mathfrak{m}_{\pi(p)}^2 \longrightarrow \operatorname{Spec} \mathcal{O}_{X,p}/\mathfrak{m}_p^2$$

of $\operatorname{Spec} \mathcal{O}_{X,p}/\mathfrak{m}_p^2 \longrightarrow \operatorname{Spec} \mathcal{O}_{S,\pi(p)}/\mathfrak{m}_{\pi(p)}^2$. This section is actually an isomorphism, as can be readily verified from the unicity in the definition of unramifiedness. We have thus proved the isomorphism

$$\mathcal{O}_{S,\pi(p)}/\mathfrak{m}_{\pi(p)}^2 \xrightarrow{\sim} \mathcal{O}_{X,p}/\mathfrak{m}_p^2.$$

Similarly, by induction we prove that

$$\mathcal{O}_{S,\pi(p)}/\mathfrak{m}_{\pi(p)}^n \xrightarrow{\sim} \mathcal{O}_{X,p}/\mathfrak{m}_p^n$$

for every $n \in \mathbf{N}$, thereby obtaining an isomorphism on the completions:

$$\hat{\mathcal{O}}_{S,\pi(p)} \xrightarrow{\sim} \hat{\mathcal{O}}_{X,p}.$$

∎

Exercise 1.8. In general (that is, no condition on the residue fields), prove that $\hat{\mathcal{O}}_{X,p}$ is a finite $\hat{\mathcal{O}}_{S,\pi(p)}$ algebra isomorphic to a finite direct sum $\bigoplus_{i=1}^n \hat{\mathcal{O}}_{S,\pi(p)}$ as an $\hat{\mathcal{O}}_{S,\pi(p)}$-module.

2 Local coordinates

Next, we introduce local coordinates in algebraic geometry.

Definition 2.1. Let X be an S-scheme and $p \in X$. An *étale S-chart* of X at p is a family $(x_1, \ldots, x_n)$ of sections of $\mathcal{O}_X$ in a neighborhood U of p such that the induced S-morphism

$$x = (x_1, \ldots, x_n) : U \longrightarrow \mathbf{A}_S^n$$

is étale.

Warning: this is non-standard terminology! When X is smooth over S, the above definition can be reformulated purely in terms of differentials. For this, we need some preparation.

Proposition 2.2 (Étaleness differential criterion)**.** *Let $\varphi : X \longrightarrow Y$ be a morphism of S-schemes locally of finite presentation and assume that X is smooth over S. Then φ is étale if and only if $\varphi^*\Omega^1_{Y/S} \longrightarrow \Omega^1_{X/S}$ is an isomorphism.*

Proof. By Theorem 1.4 and Theorem 3.7 of last lecture the only thing there is to prove is that φ is smooth if $\varphi^*\Omega^1_{Y/S} \xrightarrow{\sim} \Omega^1_{X/S}$.

We can assume $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$, and $S = \operatorname{Spec} R$. Consider a diagram of rings

$$\begin{tikzcd}
 & & B \arrow[dll, bend right, "\varphi"'] \\
\Lambda/I & \Lambda \arrow[l] & A \arrow[l, "\rho"'] \arrow[u] \\
 & & R \arrow[u]
\end{tikzcd}$$

where $I^2 = 0$ in Λ. Since X is smooth over S, there exists an R-algebra morphism $\psi : B \longrightarrow \Lambda$ reducing to φ modulo I. The idea is to modify ψ into an A-*algebra* morphism $\tilde{\psi} : B \longrightarrow \Lambda$ reducing to φ modulo I. Define

$$D : A \longrightarrow I, \qquad f \longmapsto \rho(f) - \psi(f).$$

If I is seen as a B-module via ψ (thus, also an A-module via $A \longrightarrow B$), then $D \in \mathrm{Der}_R(A, I)$. By hypothesis, there exists $\tilde{D} \in \mathrm{Der}_R(B, I)$ lifting D. We define

$$\tilde{\psi} = \psi + \tilde{D} : B \longrightarrow \Lambda.$$

It is easy to check, using that $\tilde{D}(B) \subset I$ and $I^2 = 0$, that $\tilde{\psi}$ is an A-algebra morphism reducing to φ modulo I. ■

Remark 2.3 (Smoothness differential criterion)**.** We have actually proved that under the same hypothesis of the above theorem, X is smooth over Y if and only if

$$0 \longrightarrow \varphi^*\Omega^1_{Y/S} \longrightarrow \Omega^1_{X/S} \longrightarrow \Omega^1_{X/Y} \longrightarrow 0$$

is exact and locally split.

Theorem 2.4. *Let X be a smooth S-scheme and $p \in X$. If $(x_1, \ldots, x_n)$ is a family of local sections of $\mathcal{O}_X$ in a neighborhood of p, then the following are equivalent:*

1. *$(x_1, \ldots, x_n)$ defines an étale S-chart at p;*
2. *$(dx_1, \ldots, dx_n)$ trivializes $\Omega^1_{X/S}$ in a neighborhood of p;*
3. *If $(dx_i)_p$ denotes the image of dx_i in $\Omega^1_{X/S,p}$, then $((dx_1)_p, \ldots, (dx_n)_p)$ is a basis of the $\mathcal{O}_{X,p}$-module $\Omega^1_{X/S,p}$.*
4. *If $dx_i|_p$ denotes the image of dx_i in $\Omega^1_{X/S}(p)$, then $(dx_1|_p, \ldots, dx_n|_p)$ is a basis of the k_p-vector space $\Omega^1_{X/S}(p)$, where k_p denotes the residue field of X at p.*

An étale S-chart $(x_1, \ldots, x_n)$ can also be rightfully called a "system of local coordinates".

Proof. Exercise. Hint: the equivalence $1 \iff 2$ follows from the above étaleness differential criterion; the equivalence of $2 \iff 3 \iff 4$ follows from the fact that $\Omega^1_{X/S}$ is a vector bundle combined with the next commutative algebra lemma. ■

Lemma 2.5. *Let A be a local ring with residue field k and M be a finite free A-module.*

1. *If $(m_1, \ldots, m_r)$ is a family of elements of M whose image in $M \otimes_A k$ forms a k-basis, then $(m_1, \ldots, m_r)$ is a basis of the A-module M.*
2. *If $S \subset M$ is any generating subset of M, then there exists $m_1, \ldots, m_r$ in S such that $(m_1, \ldots, m_r)$ is a basis of M.*

Proof. To prove 1, note first that M is necessarily of rank r. Indeed, since M is finite free, we have $\mathrm{rk}_A M = \dim_k M \otimes_A k$. Let $(e_1, \ldots, e_r)$ be a basis of M, and write

$$m_j = \sum_{i=1}^{r} f_{ij} e_i, \qquad f_{ij} \in A.$$

Let $d \in A$ be the determinant of $F = (f_{ij}) \in M_{r\times r}(A)$. Since the image of $(m_1, \ldots, m_r)$ in $M \otimes_A k$ is a k-basis, the image of d in k is non-zero. Since A is local, this is equivalent to $d \in A^\times$; thus F is invertible, and $(m_1, \ldots, m_r)$ is a basis of M.

We now prove 2. Since M is finite, there exists $m_1, \ldots, m_s \in S$ generating M. In particular, their image in $M \otimes_A k$ generate it as a k-vector space. Thus $s \geq r$, and up to reordering, we may may assume that the image of $(m_1, \ldots, m_r)$ in $M \otimes_A k$ is a k-basis. We conclude by an application of 1. ■

The next corollary formalizes both the intuitive notion that a smooth scheme, say over a field, should locally look like an affine space, and the algebro-geometric analog of the "local form of a submersion".

Corollary 2.6. *An S-scheme X is smooth if and only if every point of X admits an étale S-chart.*

Proof. Sufficiency follows from the following facts: smoothness is a local property on the source, every étale morphism is smooth, $\mathbf{A}^n_S \longrightarrow S$ is smooth, and composition of smooth morphisms is smooth.

Conversely, if $X \longrightarrow S$ is smooth at $p \in X$, then $\Omega^1_{X/S}$ is a vector bundle on a neighborhood of p. Thus $\Omega^1_{X/S,p}$ is a finite free $\mathcal{O}_{X,p}$-module generated by df for $f \in \mathcal{O}_{X,p}$, and conclude by an application of the above lemma. ∎

Corollary 2.7. *Let k be a field and X be a k-scheme smooth at a rational point $p \in X(k)$. Then any local system of coordinates $(x_1, \dots, x_n)$ at p induces an isomorphism*

$$\hat{\mathcal{O}}_{X,p} \cong k[\![x_1, \dots, x_n]\!].$$

Proof. If X is smooth at p, then it admits an étale chart at p. Now we just apply Proposition 1.7. ∎

Corollary 2.8. *Let k be a field and X be a smooth algebraic variety*[1] *over k. Then its connected components are irreducible. If X is connected, then*

$$\dim X = \operatorname{rk} \Omega^1_{X/k}.$$

Proof. To prove that every connected component is irreducible, it suffices to show that $\mathcal{O}_{X,p}$ is an integral domain for every closed point p of X. Working over the base change $X_p = X \otimes_k k_p$, it follows from last corollary that

$$\hat{\mathcal{O}}_{X_p,p} \cong k_p[\![x_1, \dots, x_n]\!],$$

so that $\hat{\mathcal{O}}_{X_p,p}$ is an integral domain. Since $\mathcal{O}_{X,p}$ injects into $\mathcal{O}_{X_p,p}$, which injects into $\hat{\mathcal{O}}_{X_p,p}$ (by Krull's intersection theorem), $\mathcal{O}_{X,p}$ is also an integral domain.

To prove the dimension statement, observe first that n is the number of elements of a local coordinate system at p, which coincides with $\operatorname{rk} \Omega^1_{X/k,p}$ by Theorem 2.4. On the other hand, standard properties of the Krull dimension give:

$$n = \dim k_p[\![x_1, \dots, x_n]\!] = \dim \hat{\mathcal{O}}_{X_p,p} = \dim \mathcal{O}_{X_p,p} = \dim \mathcal{O}_{X,p}$$

where the last equality follows from the fact that $\mathcal{O}_{X_p,p} = \mathcal{O}_{X,p} \otimes_k k_p$ is finite over $\mathcal{O}_{X,p}$. ∎

3 Other smoothness criteria

Theorem 3.1 (Jacobian criterion)**.** *Let $i : Z \hookrightarrow X$ be an S-immersion, and suppose that X is smooth over S. Then Z is smooth over S at $p \in Z$ if and only if the conormal sequence*

$$0 \longrightarrow C_{Z/X} \longrightarrow i^*\Omega^1_{X/S} \longrightarrow \Omega^1_{Z/S} \longrightarrow 0$$

is exact and locally split in a neighborhood of p.

[1] Here, 'algebraic variety' = 'separated of finite type'.

Proof. By last lecture, we only have to prove the necessity. We can assume that $X = \operatorname{Spec} A$, $Z = \operatorname{Spec} A/J$ for some ideal $J \subset A$, and $S = \operatorname{Spec} R$. Consider a solid diagram

$$\begin{tikzcd}
 & & A/J \arrow[ddll, bend right, "\varphi"'] \\
 & & A \arrow[u] \arrow[dl, dotted, bend right, "\psi"'] \\
\Lambda/I & \Lambda \arrow[l] & R \arrow[l] \arrow[u]
\end{tikzcd}$$

where $I^2 = 0$ in Λ. Since X is smooth over S, there exists a dotted R-algebra morphism $\psi : A \longrightarrow \Lambda$ making the diagram commute.

The idea is to modify ψ into $\tilde{\psi} : A \longrightarrow \Lambda$ such that $\tilde{\psi}(f) = 0$ for every $f \in J$. For this, we can assume that

$$\begin{tikzcd}
0 \arrow[r] & J/J^2 \arrow[r] & \Omega^1_{A/R} \otimes_R A/J \arrow[r] \arrow[l, dotted, bend right, "\langle D,-\rangle"'] & \Omega^1_{(A/J)/R} \arrow[r] & 0
\end{tikzcd}$$

is exact and split by some $D \in \operatorname{Der}_R(A, J/J^2)$ satisfying $D(f) = f + J^2$ for every $f \in J$. Since $\psi(f) \in I$ for every $f \in J$,

$$\tilde{D} : A \longrightarrow I, \qquad f \longmapsto \psi(D(f))$$

is a well defined R-derivation, where the A-module structure of I is given by ψ. We set $\tilde{\psi} = \psi - \tilde{D}$. ∎

The Jacobian criterion in its classical form easily follows.

Corollary 3.2. *Let $i : Z \hookrightarrow X$ be an S-immersion with ideal $\mathcal{I} \subset \mathcal{O}_X$, and suppose that $X \longrightarrow S$ is smooth of relative dimension n. The following are equivalent:*

1. *$Z \longrightarrow S$ is smooth at $p \in Z$, of relative dimension r.*
2. *If $(x_1, \ldots, x_n)$ is an étale S-chart of X at $i(p)$, and if $f_1, \ldots, f_N$ are local generators of $\mathcal{I}$ at $i(p)$, then up to reindexing, $\mathcal{I}$ is generated by $f_{r+1}, \ldots, f_n$ at $i(p)$ and $(x_1, \ldots, x_r, f_{r+1}, \ldots, f_n)$ defines an étale S-chart of X at $i(p)$.*
3. *There exist sections $f_{r+1}, \ldots, f_n$ of $\mathcal{O}_X$ in a neigborhood of $i(p)$ generating $\mathcal{I}$ and such that $df_{r+1}|_{i(p)}, \ldots, df_n|_{i(p)}$ are linearly independent in $\Omega^1_{X/S}(i(p))$.*

Proof. Exercise. ∎

Let us understand the relation of Jacobian matrices. With the same notation of the above corolary, let us fix an étale S-chart $(x_1, \ldots, x_n)$ of X at $i(p)$. For any section of $\mathcal{O}_X$ in a neighborhood of p we denote

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i.$$

Then Z is smooth over S at p of relative dimension r if and only if there exist generating sections $f_{r+1}, \ldots, f_n$ of $\mathcal{I}$ in a neighborhood of $i(p)$ such that

$$\operatorname{rk} \left(\frac{\partial f_i}{\partial x_j}(p) \right)_{r+1 \leq i \leq n, 1 \leq j \leq n} = n - r.$$

Finally we disccuss the relation between smoothness and flatness, and a fiberwise criterion for smoothness.

Recall that for any A-module M, the functor $M \otimes_A -$ is right exact. We say that M is *flat* if $M \otimes_A -$ is also left exact. This is equivalent to $\mathrm{Tor}_i^R(M,N) = \mathrm{Tor}_i^R(N,M) = 0$ for every A-module N and every $i \geq 1$.

For instance, if A is a field, then every A-module is flat. In general, the basic counterexample to flatness is given by $M = A/I$ where $I \subset A$ is any non-trivial ideal.

We say that a morphism of rings $A \longrightarrow B$ is flat if B is flat over A as an A-module. Finally, a morphism of schemes $\varphi : X \longrightarrow Y$ is *flat* if $\mathcal{O}_{Y,\varphi(p)} \longrightarrow \mathcal{O}_{X,p}$ is flat for every $p \in X$.

Proposition 3.3. *Every smooth morphism $\pi : X \longrightarrow S$ is flat.*

Proof. We could prove this directly (see [1] p. 53), but we take a shortcut.

Since composition of flat morphisms is flat, $\mathbf{A}_S^n \longrightarrow S$ is clearly flat, and X can be covered by étale S-scharts, we can assume that π is étale. By "reduction to Noetherian hypotheses", we can assume that S is locally Noetherian.

It follows from Exercise 1.8 that, for any $p \in X$, $\hat{\mathcal{O}}_{S,\pi(p)} \longrightarrow \hat{\mathcal{O}}_{X,p}$ is flat. This immediately implies that $\mathcal{O}_{S,\pi(p)} \longrightarrow \mathcal{O}_{X,p}$ by a general flatness criterion for local rings (see [2] Theorem 22.4). ■

A nice consequence of flatness is that a morphism that is flat and locally of finite presentation is open (see any book on algebraic geometry). In particular, every smooth morphism is open. Note that an analogous fact is true for submersions in differential geometry.

Theorem 3.4. *Let $\pi : X \longrightarrow S$ be locally of finite presentation. Then π is smooth if and only if it is flat and the fibers $X_s = X \times_S \operatorname{Spec} k_s \longrightarrow \operatorname{Spec} k_s$ are smooth for every closed point $s \in S$.*

Proof. By the above proposition we only need to prove the sufficiency. For this, up to localizing, we can assume that $S = \operatorname{Spec} R$ where R is a local ring with residue field k, and $X = \operatorname{Spec} A$ where $A = R[x_1, \ldots, x_n]/I$ for some ideal $I \subset R[x_1, \ldots, x_n]$.

Let $p \in X$ lying above the closed point of S. As $X \otimes_R k$ is smooth at p, up to considering smaller open neighborhood of p in X, it follows from the Jacobian criterion that there exist $J = (f_{r+1}, \ldots, f_r) \subset I$ such that $J \otimes_R k = I \otimes_R k$ and $df_i|_p$ are linearly independent in $\Omega^1_{\mathbf{A}_k^n/k}(p)$.

Let $B = R[x_1, \ldots, x_n]/J$, so that we have an exact sequence

$$0 \longrightarrow I/J \longrightarrow B \longrightarrow A \longrightarrow 0$$

Since $R \longrightarrow A$ is flat, we have $\mathrm{Tor}_1^R(A,k) = 0$, so that

$$0 \longrightarrow I/J \otimes_R k \longrightarrow B \otimes_R k \xrightarrow{\sim} A \otimes_R k \longrightarrow 0$$

is exact. This proves that $I/J \otimes_R k = 0$, so that $I/J = 0$ by Nakayama's lemma. Thus $X \longrightarrow S$ is smooth at p by the Jacobian criterion. ■

References

[1] Bosch, S., Lütkebohmert, W. & Raynaud, M., *Néron Models.* Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge . Band 21. Springer-Verlag 1990.

[2] H. Matsumura, *Commutative Ring Theory*, Cambridge University press (1986).