Calculus on Schemes - Lecture 4

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May 28, 2019

Contents

1 The tangent bundle

We now introduce the dual point of view on differential forms.

Definition 1.1. Let X be an S-scheme. The *tangent sheaf* of X over S is defined by

$$T_{X/S} = (\Omega^1_{X/S})^\vee := \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_{X/S}, \mathcal{O}_X).$$

Sections of $T_{X/S}$ are called *vector fields*.

One can also think of the tangent sheaf as a sheaf of derivations. If $U = \operatorname{Spec} A$ is an affine open subset in X mapping to $V = \operatorname{Spec} R$ in S, then $\Gamma(U, T_{X/S}) = \operatorname{Der}_R(A)$.

Remark 1.2. Tangent sheaves also have another piece of structure: the Lie bracket. We will come back to this in the next lecture when we talk about connections.

The tangent sheaf is coherent when X is locally of finite type over S, and it is a vector bundle when X is smooth over S (because $\Omega^1_{X/S}$ is). In this last case, we call $T_{X/S}$ the *tangent bundle* of X over S.

Example 1.3. The tangent bundle $T_{\mathbf{A}^n_R/R}$, where $\mathbf{A}^n_R = \operatorname{Spec} R[x_1, \ldots, x_n]$ is trivialized by the vector fields $\partial/\partial x_i$, for $i = 1, \ldots, n$.

Exercise 1.4. Describe the global vector fields on $\mathbf{P}^1_R$.

Now, let $\varphi : X \longrightarrow Y$ be a morphism of S-schemes. Then we have a natural morphism of $\mathcal{O}_X$-modules

$$\varphi^*\Omega^1_{Y/S} \longrightarrow \Omega^1_{X/S}$$

which induces

$$T_{X/S} \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\varphi^*\Omega^1_{Y/S}, \mathcal{O}_X). \tag{1.1}$$

There is always a natural $\mathcal{O}_X$-morphism

$$\varphi^*T_{Y/S} \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\varphi^*\Omega^1_{Y/S}, \mathcal{O}_X)$$

which is *not* an isomorphism in general. However, if Y is smooth over S, then $\Omega^1_{Y/S}$ is a vector bundle over Y, and it is easy to show that the above morphism is an isomorphism.

Definition 1.5. Let $\varphi : X \longrightarrow Y$ be a morphism of S-schemes, and assume that Y is smooth over S. The *differential* of φ is the $\mathcal{O}_X$-morphism

$$d\varphi : T_{X/S} \longrightarrow \varphi^* T_{Y/S}$$

given by (1.1) after the identification $\varphi^* T_{Y/S} \cong \mathcal{H}om_{\mathcal{O}_X}(\varphi^* \Omega^1_{Y/S}, \mathcal{O}_X)$.

The differential $d\varphi$ is also known as the "tangent map" and can be denoted by $T\varphi$, $D\varphi$, or even φ_*.

Remark 1.6. Suppose that $X = \operatorname{Spec} A$ and $S = \operatorname{Spec} R$ are affine, and assume that X is smooth over S. Let $f \in \Gamma(X, \mathcal{O}_X) = A$ be seen as a S-morphism $f : X \longrightarrow \mathbf{A}^1_S$. Since the tangent bundle $T_{\mathbf{A}^1_S/S}$ is trivial, one can see the differential of f as a morphism $df : T_{X/S} \longrightarrow \mathcal{O}_X$. This coincides with $df \in \Omega^1_{A/R}$ after the canonical identification $\Omega^1_{A/R} = \Gamma(X, T^\vee_{X/S})$.

Proposition 1.7. *Let $\varphi : X \longrightarrow Y$ be a morphism of S-schemes, and assume that Y is a smooth S-scheme. Then we have an exact sequence of $\mathcal{O}_X$-modules*

$$0 \longrightarrow T_{X/Y} \longrightarrow T_{X/S} \xrightarrow{d\varphi} \varphi^* T_{Y/S}$$

If φ is smooth (then, in particular, X is also a smooth S-scheme), then we have an exact sequence of vector bundles

$$0 \longrightarrow T_{X/Y} \longrightarrow T_{X/S} \xrightarrow{d\varphi} \varphi^* T_{Y/S} \longrightarrow 0$$

Proof. Follows by duality from the corresponding sequences for differential forms. ■

In the above situation, $T_{X/Y} = \ker d\varphi$ is also known as the "vertical subbundle" of $T_{X/S}$ for $\varphi : X \longrightarrow Y$.

Proposition 1.8. *Let $i : Z \hookrightarrow X$ be an immersion of smooth S-schemes. Then we have an exact sequence of vector bundles*

$$0 \longrightarrow T_{Z/S} \xrightarrow{di} i^* T_{X/S} \longrightarrow N_{Z/X} \longrightarrow 0$$

where $N_{Z/X} = C^\vee_{Z/X} = \mathcal{H}om_{\mathcal{O}_Z}(C_{Z/X}, \mathcal{O}_Z)$ is the normal bundle *of i.*

Proof. Again, we just dualize the conormal exact sequence. ■

Let us now briefly discuss tangent spaces of smooth algebraic varieties. Let X be a smooth algebraic variety over a field k and, to simplify, let $p \in X(k)$ be a rational point. The fiber of $T_{X/k}$ at p is by definition

$$T_{X/k}(p) = T_{X/k,p} \otimes_{\mathcal{O}_{X,p}} k_p = \operatorname{Hom}_{\mathcal{O}_{X,p}}(\Omega^1_{X/k,p}, \mathcal{O}_{X,p}) \otimes_{\mathcal{O}_{X,p}} k_p$$

where $k_p = k$ is given the structure of an $\mathcal{O}_{X,p}$-module via $f \longmapsto f(p)$. Since $\Omega^1_{X/k,p}$ is a free $\mathcal{O}_{X,p}$-module, we have

$$\operatorname{Hom}_{\mathcal{O}_{X,p}}(\Omega^1_{X/k,p}, \mathcal{O}_{X,p}) \otimes_{\mathcal{O}_{X,p}} k_p = \operatorname{Hom}_{\mathcal{O}_{X,p}}(\Omega^1_{X/k,p}, k_p) = \operatorname{Der}_k(\mathcal{O}_{X,p}, k_p).$$

Thus

$$T_{X/k}(p) = \{v \in \operatorname{Hom}_k(\mathcal{O}_{X,p}, k) \mid v(fg) = f(p)v(g) + g(p)v(f), \text{ for every } f, g \in \mathcal{O}_{X,p}\}.$$

If $\varphi : X \longrightarrow Y$ is a k-morphism of smooth algebraic varieties and $p \in X(k)$, then the differential $d\varphi$ at p is explicitly given by

$$d\varphi|_p : T_{X/k}(p) \longrightarrow T_{Y/k}(\varphi(p)), \qquad v \longmapsto v \circ \varphi^*$$

where $\varphi^* : \mathcal{O}_{Y,\varphi(p)} \longrightarrow \mathcal{O}_{X,p}$ is the natural morphism of local rings induced by φ.

Remark 1.9 (Zariski tangent space)**.** The Zariski tangent space of X at p is by definition $\mathrm{Hom}_k(\mathfrak{m}_p/\mathfrak{m}_p^2, k)$. The conormal sequence for $p : \operatorname{Spec} k \hookrightarrow X$ gives an isomorphism $\Omega^1_{X/k}(p) \cong \mathfrak{m}_p/\mathfrak{m}_p^2$. Again, using that $\Omega^1_{X/k,p}$ is a free $\mathcal{O}_{X,p}$-module, we obtain a natural isomorphism

$$T_{X/k}(p) \xrightarrow{\sim} \mathrm{Hom}_k(\mathfrak{m}_p/\mathfrak{m}_p^2, k), \qquad v \longmapsto (f + \mathfrak{m}_p^2 \longmapsto v(f))$$

The inverse of the above map associates a linear functional $\theta : \mathfrak{m}_p/\mathfrak{m}_p^2 \longrightarrow k$ to the derivation $f \longmapsto \theta(f - f(p))$.

Another point of view on tangent spaces is given by dual numbers. It is specially useful in theory of group schemes. Namely, let us denote by o the k-rational point of $\operatorname{Spec} k[\epsilon]$ given by

$$o^* : k[\epsilon] \longrightarrow k, \qquad \epsilon \longmapsto 0$$

A k-morphism of schemes $\theta : \operatorname{Spec} k[\epsilon] \longrightarrow X$ satisfying $\theta(o) = p$ corresponds to k-morphisms of algebras $\theta^* : \mathcal{O}_{X,p} \longrightarrow k[\epsilon]$ sending $\mathfrak{m}_p$ to (ϵ). Thus, θ^* is necessarily of the form

$$\theta^*(f) = f(p) + v(f)\epsilon$$

where $f(p) \in k$ is the image of f modulo $\mathfrak{m}_p$ and $v \in \mathrm{Der}_k(\mathcal{O}_{X,p}, k)$. Thus, we have a bijection

$$T_{X/k}(p) \xrightarrow{\sim} \{\theta \in \mathrm{Hom}_k(\operatorname{Spec} k[\epsilon], X) \mid \theta(o) = p\}$$

When X is an algebraic group over k, then $\mathrm{Hom}_k(\operatorname{Spec} k[\epsilon], X) = X(k[\epsilon])$ has a natural group structure, and we can prove (exercise!) that this induces the same vector space structure given by the identification with $T_{X/k}(p)$.

Example 1.10. Let $X = \mathrm{SL}_{2,\mathbf{C}}$, that is, X is the closed subscheme of $M_{2\times 2,\mathbf{C}} \cong \mathbf{A}^4_{\mathbf{C}}$ defined by the equation $\det = 1$. The Jacobian criterion shows that X is a smooth $\mathbf{C}$-scheme. Let $e \in X(\mathbf{C})$ be the identity. Then

$$T_{X/\mathbf{C}}(e) = \{A \in M_{2\times 2}(\mathbf{C}) \mid \operatorname{Tr} A = 0\}$$

Indeed, we can identify $T_{X/\mathbf{C}}(e)$ with the $\mathbf{C}$-vector space of matrices of the form

$$V = \begin{pmatrix} 1 + a\epsilon & b\epsilon \\ c\epsilon & 1 + d\epsilon \end{pmatrix}$$

such that $\det V = 1$. But since $\epsilon^2 = 0$, we have $1 = \det V = (1 + a\epsilon)(1 + d\epsilon) - (b\epsilon)(c\epsilon) = 1 + (a + d)\epsilon$, so that $a + d = 0$.

Exercise 1.11. Let $\pi : X \longrightarrow S$ be a morphism of schemes, and define the *Picard functor* $\mathrm{Pic}_{X/S}$ by

$$\mathrm{Pic}_{X/S}(T) = \mathrm{Pic}(X \times_S T)/\,\mathrm{Pic}(T) \qquad (T \in \mathsf{Sch}_{/S})$$

We say that $\pi_*\mathcal{O}_X = \mathcal{O}_S$ holds universally if $(\pi_T)_*\mathcal{O}_{X\times_S T} = \mathcal{O}_T$ for every S-scheme T. For instance, this holds if π is proper, flat, surjective and with geometrically integral fibers. Now, let $S = \operatorname{Spec} k$ where k is a field, assume that $\pi_*\mathcal{O}_X = \mathcal{O}_S$ holds universally, and that $\mathrm{Pic}_{X/k}$ is representable by a smooth k-scheme. Let $e \in \mathrm{Pic}_{X/k}(k) = \mathrm{Pic}(X)$ be given by the trivial line bundle $\mathcal{O}_X$ on X. Prove that there's a natural isomorphism of k-vector spaces

$$T_{\mathrm{Pic}_{X/k}/k}(e) = H^1(X, \mathcal{O}_X).$$

In particular, the tangent space at the origin of an elliptic curve E is naturally isomorphic to $H^1(E, \mathcal{O}_E)$. In general, if A is an abelian variety over k, and $A^\vee$ denotes the dual abelian variety, then the tangent space at the origin of $A^\vee$ is naturally isomorphic to $H^1(A, \mathcal{O}_A)$.

2 Algebraic curves

Let k be a field.

Definition 2.1. An *algebraic curve* over k is an algebraic variety over k such that all of its irreducible components have dimension 1.

Thus, $\mathbf{A}^1_k$, $\mathbf{P}^1_k$, and $\operatorname{Spec} k[x,y]/(y^2-x^3)$ are algebraic curves.

Example 2.2 (Hyperelliptic curves)**.** Let $f(x) \in k[x]$ be of degree d, and assume that over an algebraic closure $\overline{k}$ of k we have $f(x) = \prod_{i=1}^d (x-a_i)$, with $a_i \in \overline{k}$ pairwise distinct. Set

$$U \coloneqq \operatorname{Spec} k[x,y]/(y^2 - f(x))$$

and

$$V \coloneqq \begin{cases} \operatorname{Spec} k[t,s]/(s^2 - t^d f(1/t)) & d = 2e \\ \operatorname{Spec} k[t,s]/(s^2 - t^{d+1} f(1/t) & d = 2e-1 \end{cases}$$

Note that $t^d f(1/t) = \prod_{i=1}^d (1-a_i t)$. We can glue U and V via $(t,s) = (1/x, y/x^e)$ to form a scheme X over k. Note that X is smooth over k by the Jacobian criterion. Also,

$$X \setminus U = \begin{cases} \{\infty_1, \infty_2\} & d \text{ even} \\ \{\infty\} & d \text{ odd} \end{cases}$$

where ∞_1, ∞_2 are given by $(t,s) = (0, \pm 1)$ (resp. ∞ is given by $(t,s) = (0,0)$).

Let us now discuss the Riemann-Roch theorem. To keep things simple, we assume from now on that

$$X \text{ is a smooth, projective, geometrically connected curve over a field } k$$

This is the algebraic analog of a compact Riemann surface, where the original Riemann-Roch was formulated. The only caveat is that we do not assume k to be algebraically closed or to be of characteristic 0.

Recall that a *divisor* D on X is a formal finite linear combination of closed points of X with coefficients in $\mathbf{Z}$:

$$D = n_1[p_i] + \cdots + n_r[p_r]$$

where $n_i \in \mathbf{Z}$ and $p_i \in X$ is a closed point. These form an abelian group $\operatorname{Div}(X)$.

Lemma 2.3. *For any closed point $p \in X$, $\mathcal{O}_{X,p}$ is a discrete valuation ring whose uniformizers are given by local coordinates x in a neighborhood of p.*

Proof. We already know that $\mathcal{O}_{X,p}$ is a Noetherian domain; it suffices to prove that $\mathfrak{m}_p$ is principal, generated by any local coordinate. It follows from the conormal exact sequence for $p : \operatorname{Spec} k_p \longrightarrow X$ that

$$\mathfrak{m}_p/\mathfrak{m}_p^2 \cong \Omega^1_{X/k}(p)$$

By last lecture, $1 = \dim X = \operatorname{rk} \Omega^1_{X/k} = \dim_{k_p} \Omega^1_{X/k}(p) = \dim_{k_p} \mathfrak{m}_p/\mathfrak{m}_p^2$. This proves that $\mathfrak{m}_p$ is principal (Nakayama's lemma); it also follows from the above isomorphism that a generator is given by a coordinate x since $dx(p) \neq 0$. ∎

Let x be a local coordinate at p. If $f \in \operatorname{Frac} \mathcal{O}_{X,p} \setminus \{0\}$, we denote by $\operatorname{ord}_p(f) \in \mathbf{Z}$ the unique integer such that

$$f = ux^{\operatorname{ord}_p(f)}$$

for some $u \in \mathcal{O}_{X,p}^{\times}$. In particular we can define $\operatorname{ord}_p(f)$ for any rational function $f \in k(X)$ on X.

Example 2.4 (Principal divisors)**.** Let $f \in k(X)\setminus\{0\}$ be a rational function. Then the principal divisor associated to f is defined by

$$\operatorname{div}(f) = \sum_{p \in X \text{ closed}} \operatorname{ord}_p(f)[p].$$

Note that $\operatorname{ord}_p(f) = 0$ for all but finitely many closed points $p \in X$.

Locally, every divisor is a principal divisor (consider local coordinates).

We say that a divisor D is *effective*, and we denote $D \geq 0$, if $n_i \geq 0$ for every i. An effective divisor $D = \sum_p n_p[p]$ can be seen as a finite closed subscheme $D \subset X$ such that $\mathcal{I}_{D,p} = \mathfrak{m}_p^{n_p}$.

Definition 2.5. Let D be a divisor on X. We define a line bundle $\mathcal{O}_X(D)$ on X by

$$\Gamma(U, \mathcal{O}_X(D)) = \{f \in k(U) \mid \operatorname{div}(f) + D|_U \geq 0\}.$$

Note that this is indeed a line bundle. Locally, V is an open subset where D is defined by some rational function g, then

$$\mathcal{O}_X(D)|_V = g^{-1}\mathcal{O}_V.$$

We thus obtain a morphism of abelian groups $\operatorname{Div}(X) \longrightarrow \operatorname{Pic}(X)$, i.e., $\mathcal{O}_X(D_1 + D_2) = \mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)$ and $\mathcal{O}_X(-D) = \mathcal{O}_X(D)^\vee$.

Example 2.6. If D is effective, then $\mathcal{O}(-D) = \mathcal{I}_D$ is the ideal of $D \subset X$.

Now, to every line bundle L on X, we can define its degree by

$$\deg L = \chi(L) - \chi(\mathcal{O}_X).$$

On the other hand, there's an obvious notion of degree for a divisor D:

$$\deg D = n_1 \deg(p_1) + \cdots + n_r \deg(p_r)$$

where $\deg(p) = [k_p : k]$.

Theorem 2.7 (Riemann)**.** *For any divisor D on X, we have*

$$\deg \mathcal{O}_X(D) = \deg D.$$

Proof. Let us first assume that D is effective. Then we have an exact sequence

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0.$$

Tensoring with $\mathcal{O}_X(D)$, we get

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathcal{O}_X(D)|_D \longrightarrow 0.$$

Since D is a finite subscheme of X, any line bundle over D is trivial, so that $\mathcal{O}_X(D)|_D \cong \mathcal{O}_D$. Taking Euler characteristics, we obtain

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \chi(\mathcal{O}_D)$$

Since $\chi(\mathcal{O}_D) = \dim_k H^0(D, \mathcal{O}_D) = \deg D$, this proves that $\deg \mathcal{O}_X(D) = \deg D$.

If D is any divisor, we write $D = D^+ - D^-$, where D^+ and D^- are effective. We consider the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-D^-) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{D^-} \longrightarrow 0.$$

and tensor by $\mathcal{O}_X(D^+)$ to obtain

$$0 \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathcal{O}_X(D^+) \longrightarrow \mathcal{O}_X(D^+)|_{D^-} \longrightarrow 0$$

Since $\mathcal{O}_X(D^+)|_{D^-} \cong \mathcal{O}_{D^-}$, taking Euler characteristics on the above sequence gives

$$\chi(\mathcal{O}_X(D^+)) = \chi(\mathcal{O}_X(D)) + \deg D^-$$

so that

$$\deg D^+ = \chi(\mathcal{O}_X(D^+)) - \chi(\mathcal{O}_X) = \chi(\mathcal{O}_X(D)) - \chi(\mathcal{O}_X) + \deg D^- = \deg \mathcal{O}_X(D) + \deg D^-.$$

■

To get the actual Riemann-Roch formula, we use Serre duality to relate an H^1 to an H^0. We take it as a black box.

Theorem 2.8 (Serre duality)**.** *Let X be a smooth projective variety of dimension n over k. Then $\det \Omega^1_{X/k} := \bigwedge^n \omega^1_{X/k}$ is a dualising sheaf for X. In particular, for every vector bundle $\mathcal{E}$ over X and every $0 \le i \le n$, we have a canonical k-isomorphism*

$$H^i(X, \mathcal{E})^\vee = H^{n-i}(X, \mathcal{E}^\vee \otimes \det \Omega^1_{X/k}).$$

Definition 2.9. The *genus* of X is defined by $g = \dim_k H^0(X, \Omega^1_{X/k})$.

By Serre duality, we could also define $g = \dim_k H^1(X, \mathcal{O}_X)$. In particular,

$$\deg \Omega^1_{X/k} = 2g - 2.$$

Example 2.10. We've seen that $\mathbf{P}^1_k$ is of genus 0 and any elliptic curve is of genus 1.

Exercise 2.11. Let X be a hyperelliptic curve given by $y^2 = f(x)$ as before.

1. Consider the divisor
$$D = \begin{cases} e([\infty_1] + [\infty_2]) & d = 2e \\ 2e[\infty] & d = 2e - 1 \end{cases}$$
Prove thtat $(1, x, x^2, \ldots, x^e, y)$ is a basis of $H^0(X, \mathcal{O}_X(D))$. Conclude that X is projective.

2. Prove that
$$\left(\frac{dx}{y}, x\frac{dx}{y}, \ldots, x^{e-2}\frac{dx}{y}\right)$$
is a basis of $H^0(X, \Omega^1_{X/k})$. This shows that X is of genus $e - 1$.

Theorem 2.12 (Riemann-Roch)**.** *For every divisor D on X, we have*

$$\dim_k H^0(X, \mathcal{O}_X(D)) - \dim_k H^0(X, \mathcal{O}_X(-D) \otimes \Omega^1_{X/k}) = \deg D + 1 - g.$$

Proof. We have

$$\begin{aligned}\deg \mathcal{O}_X(D) &= \chi(\mathcal{O}_X(D)) - \chi(\mathcal{O}_X) \\ &= (\dim_k H^0(X, \mathcal{O}_X(D)) - \dim_k H^1(X, \mathcal{O}_X(D))) - (\dim_k H^0(X, \mathcal{O}_X) - \dim_k H^1(X, \mathcal{O}_X)) \\ &= (\dim_k H^0(X, \mathcal{O}_X(D)) - \dim_k H^0(X, \mathcal{O}_X(-D) \otimes \Omega^1_{X/k})) - (1 - \dim_k H^0(X, \Omega^1_{X/k}))\end{aligned}$$

Now we just apply Riemann's theorem. ■