Calculus on Schemes - Lecture 5

Tiago J. Fonseca

June 04, 2019

Contents

1 Algebraic connections

Definition 1.1. Let $\pi : X \longrightarrow S$ be an S-scheme and $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_X$-module. A *connection* on $\mathcal{E}$ is an $\pi^{-1}\mathcal{O}_S$-morphism

$$\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}$$

such that

$$\nabla(fe) = e \otimes df + f\nabla e$$

for any local sections f of $\mathcal{O}_X$ and e of $\mathcal{E}$. The couple $(\mathcal{E}, \nabla)$ is a *module with connection.* A local section e of $\mathcal{E}$ such that $\nabla e = 0$ is said to be *horizontal* (or *flat*).

Note that a connection is not $\mathcal{O}_X$-linear! If $\mathcal{E}$ is a vector bundle, we also say that $(\mathcal{E}, \nabla)$ is a vector bundle with connection.

Example 1.2 (Trivial bundle)**.** Suppose that $\mathcal{E} = \mathcal{O}_X e_1 \oplus \cdots \oplus \mathcal{O}_X e_r$ is a vector bundle of rank r trivialized by $(e_1, \ldots, e_r)$. Let $\omega_{ij} \in \Gamma(X, \Omega^1_{X/S})$ be defined by

$$\nabla e_j = \sum_{i=1}^{r} e_i \otimes \omega_{ij}, \qquad j = 1, \ldots, r.$$

The matrix of differential forms $\Omega = (\omega_{ij})$ is classically known as the *matrix of the connection*; it completely determines ∇, which can be described symbolically as

$$\nabla = d + \Omega.$$

In fact, if

$$e = \sum_{j=1}^{r} f_j e_j$$

is any section of $\mathcal{E}$, then

$$\nabla e = \sum_{i=1}^{r} e_i \otimes \left(df_i + \sum_{j=1}^{r} \omega_{ij} f_j \right).$$

In particular, horizontal sections correspond to solutions $(f_1, \dots, f_r)$ of the *linear differential equation*

$$df_i + \sum_{j=1}^{r} \omega_{ij} f_j = 0, \qquad i = 1, \dots, r.$$

A connection $\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}$ induces an $\mathcal{O}_X$-morphism

$$T_{X/S} \longrightarrow \mathcal{E}nd_{\mathcal{O}_S}(\mathcal{E}), \qquad v \longmapsto \nabla_v$$

where

$$\nabla_v(e) = \langle v, \nabla e \rangle$$

is the *covariant derivative of* e *along* v. Note that, for every sections f of $\mathcal{O}_X$ and e of $\mathcal{E}$,

$$\nabla_v(fe) = v(f)e + f\nabla_v(e)$$

where $v(f) = \langle v, df \rangle$.

Example 1.3. Let k be a field. To every homogeneous ODE

$$g_n \frac{d^n y}{dx^n} + g_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + g_1 \frac{dy}{dx} + g_0 y = 0 \tag{E}$$

where $g_i \in k[x]$, we can associate a vector bundle with connection $(\mathcal{E}, \nabla)$ as follows. Let $X = D(g_n) \subset \mathbf{A}^1_k$ be the open subset of the affine line over which $g_n \neq 0$, and consider the trivial bundle $\mathcal{E} = \mathcal{O}_X^{\oplus n}$. Then we define

$$\nabla = d + \Omega : \mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega^1_{X/k}$$

by

$$\Omega = \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{g_0}{g_n} & \frac{g_1}{g_n} & \frac{g_2}{g_n} & \cdots & \frac{g_{n-1}}{g_n} \end{pmatrix} dx$$

With this definition, solutions of (E) (when they exist) correspond to horizontal sections of ∇. Indeed, if y is a solution of (E), then

$$e = \left(y, \frac{dy}{dx}, \dots, \frac{d^{n-1}y}{dx^{n-1}} \right)$$

satisfies $\nabla_{\frac{d}{dx}} e = 0$. Since $T_{X/k}$ is trivialized by $\frac{d}{dx}$, this means that e is horizontal: $\nabla e = 0$. Conversely, every horizontal section of ∇ must be of the above form.

Remark 1.4. Let us take $k = \mathbf{Q}$. One may ask the following question: when does (E) has n linearly independent *algebraic* solutions $y_i \in \overline{\mathbf{Q}(x)}$ (= an algebraic closure of the function field $\mathbf{Q}(x)$). There's a conjecture of Grothendieck (also known as 'Grothendieck-Katz p-curvature conjecture') which says that this should happen if and only if it does modulo p for almost all prime numbers p.

We can perform a number of natural multilinear operations on connections. Let $(\mathcal{E}_1, \nabla_1)$ and $(\mathcal{E}_2, \nabla_2)$ be modules with connection

1. We define a connection ∇ on $\mathcal{E}_1 \oplus \mathcal{E}_2$ by

$$\nabla_v(e_1 + e_2) = \nabla_{1,v}(e_1) + \nabla_{2,v}(e_2)$$

2. We define a connection ∇ on $\mathcal{E}_1 \otimes_{\mathcal{O}_X} \mathcal{E}_2$ by

$$\nabla_v(e_1 \otimes e_2) = \nabla_{1,v}(e_1) \otimes e_2 + e_1 \otimes \nabla_{2,v}(e_2)$$

3. We define a connection ∇ on $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}_1, \mathcal{E}_2)$ by

$$\nabla_v(\varphi)(e_1) = \nabla_{2,v}(\varphi(e_1)) - \varphi(\nabla_{1,v}(e_1))$$

A morphism $(\mathcal{E}_1, \nabla_1) \longrightarrow (\mathcal{E}_2, \nabla_2)$ of modules with connection is a horizontal global section of $(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}_1, \mathcal{E}_2), \nabla)$. We thus obtain a category of modules with connection.

In particular, if $\mathcal{O}_X$ is taken with the trivial connection d, we obtain the dual $(\mathcal{E}^\vee, \nabla^\vee)$ of $(\mathcal{E}, \nabla)$ via

$$\nabla_v^\vee(\varphi)(e) = v(\varphi(e)) - \varphi(\nabla_v(e))$$

Exercise 1.5. Let $(\mathcal{E}, \nabla)$ be a module with connection. Describe the induced connection on the symmetric algebra $\operatorname{Sym}_{\mathcal{O}_X} \mathcal{E}$.

Let us now define integrability. In an analytic context (see following section), this notion is linked with existence of horizontal sections, which is equivalent to the solvability of certain linear differential equations. Nonetheless, integrability can be defined in a purely algebraic context.

Recall that the tangent sheaf $T_{X/S} = (\Omega_{X/S})^\vee$ of $\pi : X \longrightarrow S$ can also be seen as a sheaf of $\pi^{-1}\mathcal{O}_S$-linear derivations of $\mathcal{O}_X$. In particular, we can consider the commutator in $\mathcal{E}nd_{\pi^{-1}\mathcal{O}_s}(\mathcal{O}_X)$ of vector fields:

$$[v_1, v_2] = v_1 \circ v_2 - v_2 \circ v_1$$

and one may easily verify that $[v_1, v_2]$ is again a section of $T_{X/S} \subset \mathcal{E}nd_{\pi^{-1}\mathcal{O}_s}(\mathcal{O}_X)$. This gives $T_{X/S}$ the structure of a sheaf of Lie algebras.

Example 1.6. On $T_{\mathbf{A}^1_R/R}$, we have $[f\frac{d}{dx}, g\frac{d}{dx}] = \frac{d(fg)}{dx}\frac{d}{dx}$.

Definition 1.7. We say that a connection $\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega^1_{X/S}$ is *integrable* if

$$\nabla_{[v_1,v_2]} = \nabla_{v_1} \circ \nabla_{v_2} - \nabla_{v_2} \circ \nabla_{v_1}$$

for every vector fields v_1 and v_2.

Exercise 1.8. Prove that if X is smooth of relative dimension 1 over S, then every connection if integrable. (Exercise!)

Remark 1.9. For general culture, let us understand why horizontal sections have this name. Let $\mathcal{E}$ be a vector bundle on a smooth S-scheme X, and denote by $p : E = \mathbf{V}(\mathcal{E}^\vee) \longrightarrow X$ the associtated space, so that sections of p correspond to sections of $\mathcal{E}$. Since p is smooth, we have an exact sequence

$$0 \longrightarrow T_{E/X} \longrightarrow T_{E/S} \xrightarrow{dp} p^*T_{X/S} \longrightarrow 0.$$

One can show that any connection $\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega^1_{X/S}$ induces a splitting

$$T_{E/S} \cong T_{E/X} \oplus \mathcal{H}_\nabla.$$

The subvector bundle $\mathcal{H}_\nabla \subset T_{E/S}$ over E is the *horizontal subbundle* given by ∇. It is related to ∇ as follows: a local section e of $\mathcal{E}$ over U, which can also be seen as a section $e : U \longrightarrow E$ of p, is horizontal for ∇ if and only if

$$de : T_{U/S} \longrightarrow e^*T_{E/S}$$

factors through $e^*\mathcal{H}_\nabla$. The reason $\mathcal{H}_\nabla$ is called the horizontal subbundle is because it is complementary to the vertical direction given by $T_{E/X}$ (picture the morphism p vertically). Note that ∇ is integrable if and only if $\mathcal{H}_\nabla$ is an "integrable subbundle of $T_{E/S}$", i.e., $[\mathcal{H}_\nabla, \mathcal{H}_\nabla] \subset \mathcal{H}_\nabla$.

2 Analytic connections and local systems

Let M be a complex manifold.

Definition 2.1. Let A be a ring. An *A-local system* on M is a locally constant sheaf of A-modules on M.

The category of A-local systems is the full subcategory of sheaves of A-modules on M.

Let $\mathcal{L}$ be an A-local system on M, and fix a point $p \in M$. For any continous loop $\gamma : [0,1] \longrightarrow M$ at p, we can cover $\gamma([0,1])$ by open subsets $U_0, U_1, \ldots, U_n \subset M$ such that

1. $\mathcal{L}$ is constant over each U_i
2. $U_i \cap U_{i+1} \neq \emptyset$ for every $0 \leq i \leq n-1$
3. $p \in U_0 \cap U_n$

By choosing points $p_i \in U_i \cap U_{i+1}$, we obtain an A-automorphism T_γ of $\mathcal{L}_p$ by composing the sequence of isomorphisms

$$\mathcal{L}_p \xrightarrow{\sim} \mathcal{L}_{p_1} \xrightarrow{\sim} \cdots \xrightarrow{\sim} \mathcal{L}_{p_{n-1}} \xrightarrow{\sim} \mathcal{L}_p$$

One can check that T_γ does not depend on the choice of U_i, neither of p_i. Also, it doesn't depend on the choice of homotopy class of γ, and is compatible with path composition. We thus get a representation

$$\pi_1(M,p) \longrightarrow \mathrm{Aut}_A(\mathcal{L}_p)$$

This is the *monodromy* at p.

Lemma 2.2. *Suppose that M is connected, and let $p \in M$. The monodromy at p defines an equivalence between the category of A-local systems on M and the category of A-linear representations of $\pi_1(M,p)$.*

Proof. Let $\mathcal{L}$ be a sheaf and $q : L \longrightarrow M$ be the étalé space of $\mathcal{L}$, i.e., $q^{-1}(p) = \mathcal{L}_p$ for every $p \in M$, and the topology of L is generated by the open subsets $[U,s] = \{s_p \in F \mid p \in U\}$, where $s \in \Gamma(U, \mathcal{L})$. Then $\mathcal{L}$ is locally constant if and only if $q : L \longrightarrow M$ is a covering space of M. We then use that covering spaces correspond to π_1-sets. We leave the details as an exercise. ■

Example 2.3. Let $\alpha \in \mathbf{C}$, and consider the ODE on $\mathbf{C}^\times$:

$$z\frac{dy}{dz} + \alpha y = 0 \tag{E}$$

Then

$$U \longmapsto \mathcal{L}_E(U) \coloneqq \{y \in \Gamma(U, \mathcal{O}_U) \mid y \text{ is a solution of (E)}\}$$

is a subsheaf of the sheaf of holomorphic functions $\mathcal{O}_{\mathbf{C}^\times}$. For $U \subset \mathbf{C}^\times$ simply connected, every solution y of (E) is of the form

$$y = \lambda z^{-\alpha} := \lambda e^{-\alpha \log z}$$

where log is some fixed determination of the logarithm in U, and $\lambda \in \mathbf{C}$. Thus $\mathcal{L}_E$ is a **C**-local system (of rank 1).

Exercise 2.4. Let $\gamma \in \pi_1(\mathbf{C}^\times, 1)$ be given by $t \longmapsto e^{2\pi i t}$. Prove that the monodromy at 1 of the above local system $\mathcal{L}_E$ sends γ to the multiplication by $e^{-2\pi i \alpha}$.

The above example generalizes to other differential equations. We define connections on vector bundles over complex manifolds in the same way we do for schemes. These are classically known as "holomorphic connections".

Proposition 2.5. *Let $(\mathcal{E}, \nabla)$ be a vector bundle with connection on a complex manifold M, and denote by $\mathcal{E}^\nabla$ the subsheaf of $\mathcal{E}$ given by horizontal sections. Then:*

1. *For every $p \in M$, the natural map $\mathcal{E}_p^\nabla \longrightarrow \mathcal{E}(p)$ is injective. In particular, the stalk $\mathcal{E}_p^\nabla$ is finite-dimensional for every $p \in M$.*
2. *If $p \longmapsto \dim_{\mathbf{C}} \mathcal{E}_p^\nabla$ is locally constant, then $\mathcal{E}^\nabla$ is a **C**-local system.*

Proof. Exercise. Hint: use unicity of solutions of ODEs with given initial condition. ■

The above proposition also has a converse. Loosely speaking, when $(\mathcal{E}, \nabla)$ is integrable, then we can "recover" $(\mathcal{E}, \nabla)$ from $\mathcal{E}^\nabla$.

Theorem 2.6 (Riemann-Hilbert correspondence)**.** *Let M be a complex manifold. The functor*

$$(\mathcal{E}, \nabla) \longmapsto \mathcal{E}^\nabla$$

*from the category of vector bundles with integrable connection on M to the category of **C**-local systems of finite rank is an equivalence of categories with inverse*

$$\mathcal{L} \longmapsto (\mathcal{L} \otimes_{\mathbf{C}} \mathcal{O}_M, \mathrm{id} \otimes d).$$

In particular, if $\dim_{\mathbf{C}} \mathcal{E}_p^\nabla = \mathrm{rk}_{\mathcal{O}_{M,p}} \mathcal{E}_p$ for every $p \in M$.

Proof. Exercise. Hint: use Frobenius' theorem (which also holds in a complex-analytic setting). ■

3 GAGA

Next, we merely sketch the main definitions and results concerning the comparison between algebraic geometry and complex analytic geometry.

Definition 3.1. A *complex analytic space* is a Hausdorff **C**-locally ringed space $(X, \mathcal{O}_X)$ locally isomorphic to

$$(V(I), \mathcal{O}_U / I)$$

where $U \subset \mathbf{C}^n$ is an open subset for the usual "euclidean" topology, $\mathcal{O}_U$ is the sheaf of *holomorphic* functions on U, I is an ideal of finite type (i.e., locally of the form $(f_1, \ldots, f_m)$), and $V(I) \subset U$ is the support of $\mathcal{O}_U / I$.

For instance, every complex manifold is a complex analytic space (locally isomorphic to $(U, \mathcal{O}_U)$ for $U \subset \mathbf{C}^n$ an open subset).

Example 3.2. Let $f_1, \dots, f_m \in \mathbf{C}[x_1, \dots, x_n]$ and $X = \operatorname{Spec} \mathbf{C}[x_1, \dots, x_n]/(f_1, \dots, f_m)$. Since any polynomial defines a holomorphic function, we can also define an analytic space X^{an} as $(V(I), \mathcal{O}_{\mathbf{C}^n}/I)$, where $I = (f_1, \dots, f_m)$, seen as holomorphic functions. Note that, as a set, $X^{\mathrm{an}} = V(I) = \{z \in \mathbf{C}^n \mid f_i(z) = 0\} = X(\mathbf{C})$.

Similarly, since morphisms of affine $\mathbf{C}$-schemes of finite type are given by polynomials, any $\mathbf{C}$-morphism $X \longrightarrow Y$ induces a morphism of analytic spaces $X^{\mathrm{an}} \longrightarrow Y^{\mathrm{an}}$.

By gluing, we obtain a functor

$$X \longmapsto X^{\mathrm{an}}$$

from the category of separated $\mathbf{C}$-schemes locally of finite type to the category of analytic spaces. Formally, X^{an} is an analytic space representing the functor

$$\mathcal{X} \longmapsto \operatorname{Hom}_{\mathbf{C}}(\mathcal{X}, X)$$

where $\operatorname{Hom}_{\mathbf{C}}$ means "morphisms of $\mathbf{C}$-locally ringed spaces".

This implies in particular that X^{an} is in natural bijection with $X(\mathbf{C})$ as a set. Moreover, one can prove that the canonical morphism $X^{\mathrm{an}} \longrightarrow X$ induces isomorphisms on the level of completed local rings: for every $x \in X$ closed,

$$\hat{\mathcal{O}}_{X,x} \xrightarrow{\sim} \hat{\mathcal{O}}_{X^{\mathrm{an}},x}.$$

Proposition 3.3. *X is reduced (resp. irreducible, resp. connected) if and only if X^{an} is reduced (resp. irreducible, resp. connected).*

Note that the permanence of irreducibility and of connecteness is not trivial since the topology on X^{an} is much finer than the Zariski topology.

Proposition 3.4. *Let (P) be one of the following properties for morphisms: smooth, étale, unramified, (open) immersion, proper, finite. Then a morphism of separated $\mathbf{C}$-schemes locally of finite type $\varphi : X \longrightarrow Y$ satisfies (P) if and only if φ^{an} satisfies (P).*

Let us remark that, for complex manifolds, the "inverse function theorem" implies that étale morphisms are local isomorphisms. An immediate corollary of the above proposition is that a smooth and proper $\mathbf{C}$-scheme X gives a compact complex manifold X^{an}.

We now discuss Serre's GAGA theorems. Let X be a separated $\mathbf{C}$-scheme locally of finite type. To any $\mathcal{O}_X$-module $\mathcal{F}$ on X, we can associate a $\mathcal{O}_{X^{\mathrm{an}}}$-module

$$\mathcal{F}^{\mathrm{an}} := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X^{\mathrm{an}}}$$

on X^{an}. A theorem of Oka asserts that the structural sheaf $\mathcal{O}_{X^{\mathrm{an}}}$ is coherent. In particular, if $\mathcal{F}$ is coherent, so is $\mathcal{F}^{\mathrm{an}}$.

Theorem 3.5 (Serre)**.** *Let X be a* projective *$\mathbf{C}$-scheme.*

1. *If $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, then there natural map*

   $$H^q(X, \mathcal{F}) \longrightarrow H^q(X^{\mathrm{an}}, \mathcal{F}^{\mathrm{an}})$$

   is an isomorphism for every $q \geq 0$.

2. *The functor $\mathcal{F} \longmapsto \mathcal{F}^{\mathrm{an}}$ is an equivalence between the category of coherent $\mathcal{O}_X$-modules and the category of coherent $\mathcal{O}_{X^{\mathrm{an}}}$-modules.*

Here are some corollaries of Serre's theorems, the proof of which we leave as an exercise:

1. (Chow's theorem) Any closed analytic subvariety of $\mathbf{P}^n(\mathbf{C})$ is algebraic (i.e., is isomorphic to the analytification of an algebraic variety).

2. If X and Y are projective $\mathbf{C}$-schemes, then any morphism of complex analytic spaces $X^{\mathrm{an}} \longrightarrow Y^{\mathrm{an}}$ comes from a morphism of $\mathbf{C}$-schemes $X \longrightarrow Y$.

3. If X is a projective $\mathbf{C}$-scheme, and $\mathcal{F}$ is a vector bundle (= locally free sheaf of finite rank) over X^{an}, then there exists a vector bundle $\mathcal{E}$ over X such that $\mathcal{E}^{\mathrm{an}} \cong \mathcal{F}$.

4. Let X be a smooth, projective, and connected $\mathbf{C}$-scheme. Then the category of vector bundles with integrable connection on X is equivalent to the category of local systems on X^{an}.

Note that the last assertion is a bit surprising, since it says that local systems on X^{an}, which are purely topological gadgets on X^{an}, can be described algebraically.

We end our discussion of GAGA with a counter-example showing that *projectiveness* is essential.

Example 3.6. Let C be a smooth, projective, and connected curve of genus g over $\mathbf{C}$, and fix a closed point $p \in C$. One can prove that the moduli of line bundles with connection on C is representable by a (smooth) $\mathbf{C}$-scheme which we denote by $\mathrm{Pic}^{\natural}(C)$. It sits on an exact sequence of $\mathbf{C}$-group schemes

$$0 \longrightarrow H^0(C, \Omega^1_{C/\mathbf{C}}) \longrightarrow \mathrm{Pic}^{\natural}(C) \longrightarrow \mathrm{Pic}^0(C) \longrightarrow 0$$

where $H^0(C, \Omega^1_{C/\mathbf{C}})$ is seen as a vector group.

On the other hand, a finite presentation of $\pi_1(C^{\mathrm{an}}, p)$ gives $\mathrm{Hom}(\pi_1(C^{\mathrm{an}}, p), \mathbf{C}^\times)$ a natural structure of $\mathbf{C}$-scheme, which we denote by T. Actually, we have an isomorphism $T \cong \mathbf{G}_m^{2g}$. Now, it follows from the Riemann-Hilbert correspondence (actually, one must show that this correspondence "varies analytically" on parameters), that there is an isomorphism

$$\mathrm{Pic}^{\natural}(C)^{\mathrm{an}} \cong T^{\mathrm{an}}$$

in the category of complex analytic spaces. But $\mathrm{Pic}^{\natural}(C)$ is *not* isomorphic to T in the category of $\mathbf{C}$-schemes. In fact, it follows from the above exact sequence that there is non-constant morphism $\mathrm{Pic}^{\natural}(C) \longrightarrow T$!