Calculus on Schemes - Lecture 6

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Contents

1 Local systems of geometric origin **1**

2 Algebraic de Rham cohomology 1 **2**

3 Hypercohomology **5**

1 Local systems of geometric origin

To every proper morphism $\pi : \mathcal{X} \longrightarrow M$ of complex manifolds, every $n \in \mathbf{N}$, and every ring A, we can associate the sheaf $R^n\pi_* A_{\mathcal{X}}$ on M. This can be shown to be the sheafification of

$$U \longmapsto H^n(\pi^{-1}(U), A),$$

where $H^n(\cdot, A)$ denotes the usual singular (or Betti) cohomology for topological spaces with coefficients in A. Most popular choices of A include: $A = \mathbf{Z}$, $A = \mathbf{Q}$, and $A = \mathbf{C}$. Its stalk at $p \in M$ is

$$(R^n\pi_* A_{\mathcal{X}})_p = H^n(\mathcal{X}_p, A)$$

where $\mathcal{X}_p = \pi^{-1}(p)$.

Lemma 1.1. *Let $\mathcal{F}$ be an abelian presheaf over M for which there exists a basis $\mathcal{B}$ of open subsets of M satisfying the following property:*

$$\text{if } U, V \in \mathcal{B},\ U \supset V, \text{ then the restriction } \mathcal{F}(U) \longrightarrow \mathcal{F}(V) \text{ is an isomorphism} \qquad \text{(C)}$$

Then the sheafification of $\mathcal{F}$ is locally constant.

Proof. Since the sheafification of $\mathcal{F}$ is the sheaf of sections of the étalé space $q : F \longrightarrow M$, we want to prove that q is a covering map. For this, it is sufficient to prove that, for every $U \in \mathcal{B}$, $q^{-1}(U)$ is the disjoint union of $[s, U] = \{s_p \in F \mid p \in U\}$ for every $s \in \mathcal{F}(U)$.

For $s, t \in \mathcal{F}(U)$, if $[s, U] \cap [t, U] \neq \emptyset$, then there exists $V \in \mathcal{B}$ such that $s|_V = t|_V$, so that $s = t$ by (C). This proves that the union $\bigcup_s [s, U] \subset q^{-1}(U)$ is disjoint. Now, if $f \in q^{-1}(U)$, then there exists $V \in \mathcal{B}$ and $s \in \mathcal{F}(V)$ such that $s_{q(f)} = f$. By (C), we can extend s to a section of $\mathcal{F}$ over U, so that $f \in [s, U]$. ∎

Proposition 1.2. *If π is surjective, proper and submersive then $R^n\pi_* A_{\mathcal{X}}$ is an A-local system on M. Its stalks are singular cohomology groups modulo torsion.*

Proof. By Ehresmann's fibration theorem, π is a locally trivial fibration of *smooth* manifolds. This means that for every $p \in M$ there exists an open neighborhood $U \subset M$ of p such that

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow[\sim]{C^\infty \text{ isom}} & \mathcal{X}_p \times U \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_2} & \\ & U & \end{array}$$

Now we apply last lemma. ■

Remark 1.3. Dually, we also have local systems given by *homology*: $R_n\pi_*A_{\mathcal{X}} := (R^n\pi_*A_{\mathcal{X}})^\vee$.

Example 1.4 (Analytic Tate curve)**.** Recall that a complex torus X is a Riemann surface isomorphic to $\mathbf{C}/\Lambda$, where $\Lambda \subset \mathbf{C}$ is a lattice (discrete cocompact subgroup). Under such presentation, there's a canonical identification $H_1(X, \mathbf{Z}) \cong \Lambda$.

One can always choose Λ to be of the form $\Lambda = \mathbf{Z} + \mathbf{Z}\tau$, for some $\tau \in \mathbf{C}$ satisfying $\operatorname{Im}\tau > 0$. We can thus describe a torus multiplicatively as follows: if $q = e^{2\pi i\tau}$, then

$$\mathbf{C}/\Lambda \xrightarrow{\sim} \mathbf{C}^\times/q^{\mathbf{Z}}, \qquad z \longmapsto e^{2\pi i z}.$$

Let $D^* = \{z \in \mathbf{C} \mid 0 < |z| < 1\}$, and define an action of $\mathbf{Z}$ on $\mathbf{C}^\times \times D^*$ by

$$n \cdot (z, q) = (zq^n, q).$$

Such action is proper and free, and we denote by $\mathcal{X}$ the quotient complex manifold. It comes equiped with a natural map $\pi : \mathcal{X} \longrightarrow D^*$ whose fiber at $q \in D^*$ is the complex torus $\mathcal{X}_q = \mathbf{C}^\times/q^{\mathbf{Z}}$.

Let us consider the local system $R_1\pi_*\mathbf{Z}_{\mathcal{X}}$. Its fiber at $q \in D^*$ is $H_1(\mathbf{C}^\times/q^{\mathbf{Z}}, \mathbf{Z})$, which we can identify to the lattice $\mathbf{Z} + \mathbf{Z}\tau \subset \mathbf{C}$, where τ is any choice of complex number with positive imaginary part satisfying $e^{2\pi i\tau} = q$. Under this identification, the monodromy action of $\pi_1(D^*, q) \cong \mathbf{Z}$ on $\operatorname{Aut}_{\mathbf{Z}}(H_1(\mathbf{C}^\times/q^{\mathbf{Z}}, \mathbf{Z}))$ is generated by

$$1 \longmapsto 1, \ \tau \longmapsto \tau + 1.$$

Now, if $\pi : \mathcal{X} \longrightarrow M$ is the analytification of a proper smooth family $X \longrightarrow S$ of $\mathbf{C}$-schemes, and S is projective, it follows from the Riemann-Hilbert correspondence, and from GAGA, that there exists an *algebraic* vector bundle with integrable connection $(\mathcal{E}, \nabla)$ over S whose analytification coincides with $(R^n\pi_*\mathbf{C}_{X^{\mathrm{an}}} \otimes_{\mathbf{C}} \mathcal{O}_{S^{\mathrm{an}}}, \mathrm{id} \otimes d)$. What is this vector bundle $\mathcal{E}$? What if S is not projective?

2 Algebraic de Rham cohomology 1

Let X be an S-scheme. For every $p \geq 0$, we define

$$\Omega^p_{X/S} = \bigwedge^p \Omega^1_{X/S}.$$

This is a quasi-coherent sheaf over S which is locally generated by sections of the form

$$g df_1 \wedge \cdots \wedge df_p.$$

If X is smooth of relative dimension n over S, then $\Omega^p_{X/S}$ is a vector bundle over X of rank $\binom{n}{p}$.

Lemma 2.1. *There's a unique family of morphisms of abelian sheaves $d : \Omega^p_{X/S} \longrightarrow \Omega^{p+1}_{X/S}$ such that*

1. *$d : \mathcal{O}_X \longrightarrow \Omega^1_{X/S}$ is the usual differential.*
2. *If ω (resp. η) is a section of $\Omega^p_{X/S}$ (resp. $\Omega^q_{X/S}$), then*
$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta$$
3. *$d^2 = 0$*

Proof. Exercise. ∎

We thus obtain a complex of quasi-coherent $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \Omega^1_{X/S} \longrightarrow \Omega^2_{X/S} \longrightarrow \cdots,$$

the *de Rham complex* of X over S. Note that if X is smooth over S, of relative dimension n, then $\Omega^p_{X/S} = 0$ for every $p > n$, so that we have the bounded complex of vector bundles

$$\Omega^\bullet_{X/S} : 0 \longrightarrow \mathcal{O}_X \longrightarrow \Omega^1_{X/S} \longrightarrow \cdots \longrightarrow \Omega^n_{X/S} \longrightarrow 0.$$

Grothendieck realized that we can use the above complex to define a reasonable cohomology theory, at least in the presence of good conditions such as smoothness and characteristic 0.

Let us start with a very simple setting.

Definition 2.2. Let k be a field and $X = \operatorname{Spec} A$ be a smooth *affine* scheme over k. The *algebraic de Rham cohomology of X over k* is the cohomology of the complex $\Omega^\bullet_{A/k}$, i.e.,

$$H^n_{\mathrm{dR}}(X/k) := \frac{\ker(d : \Omega^n_{A/k} \longrightarrow \Omega^{n+1}_{A/k})}{\operatorname{im}(d : \Omega^{n-1}_{A/k} \longrightarrow \Omega^n_{A/k})} = \frac{\text{closed } n\text{-forms}}{\text{exact } n\text{-forms}}.$$

This mimics the usual recipe for computing the de Rham cohomology of a C^∞ manifold, but here we are only dealing with algebraic differential forms.

Example 2.3. The de Rham cohomology of $\mathbf{A}^1_k$ is the cohomology of the complex

$$0 \longrightarrow k[x] \xrightarrow{d} k[x]dx \longrightarrow 0$$
$$\sum_{i\geq 0} a_i x^i \longmapsto \left(\sum_{i\geq 1} i a_i x^{i-1}\right) dx$$

Since there are no differential forms of degree ≥ 2, we have $H^n_{\mathrm{dR}}(\mathbf{A}^1_k/k) = 0$ for every $n \geq 2$. In degrees 0 and 1, we have

$$H^0_{\mathrm{dR}}(\mathbf{A}^1_k/k) = \{f \in k[x] \mid df = 0\} \text{ and } H^1_{\mathrm{dR}}(\mathbf{A}^1_k/k) = k[x]dx/\{df \mid f \in k[x]\}.$$

1. If k is of characteristic 0, then $df = 0$ if and only if $f = a_0 \in k$, so that
$$H^0_{\mathrm{dR}}(\mathbf{A}^1_k/k) = k.$$
Moreover, every $\omega = \left(\sum_{i\geq 0} a_i x^i\right) \in k[x]dx$ has a primitive, namely $F = \sum_{i\geq 0} \frac{a_i}{i+1} x^{i+1}$. Thus
$$H^1_{\mathrm{dR}}(\mathbf{A}^1_k/k) = 0.$$

2. If k is of positive characteristic p, strange things can happen. First of all, note that the derivative of x^p is 0 but x^p is not in k! In fact,

$$H^0_{\mathrm{dR}}(\mathbf{A}^1_k/k) = k[x^p].$$

For the same reason, $x^{p-1}dx$ doesn't have a primitive! In fact,

$$H^1_{\mathrm{dR}}(\mathbf{A}^1_k/k) = \bigoplus_{n\geq 1} kx^{np-1}dx$$

This example illustrates that algebraic de Rham cohomology is in general (but not always!) pathological in positive characteristic. Next we restrict our attention to fields of characteristic 0.

Exercise 2.4. Let k be a field of characteristic 0. Compute the de Rham cohomology of $\mathbf{A}^n_k$ for any $n \geq 1$. In general, prove the following homotopy invariance property: for every smooth affine scheme X over k and every $n \in \mathbf{N}$, we have $H^n_{\mathrm{dR}}(X \times_k \mathbf{A}^1_k/k) = H^n_{\mathrm{dR}}(X/k)$.

Example 2.5. Let k be a field of characteristic 0, and consider $\mathbf{G}_{m,k} = \mathbf{A}^1_k \backslash \{0\} = \operatorname{Spec} k[x, x^{-1}]$. We must compute the cohomology of the complex

$$\begin{gathered} 0 \longrightarrow k[x,x^{-1}] \longrightarrow k[x,x^{-1}]dx \longrightarrow 0 \\ \sum_i a_i x^i \longmapsto \left(\sum_i i a_i x^{i-1}\right) dx \end{gathered}$$

Note that here i is allowed to be negative. Now, the differential form $dx/x \in k[x, x^{-1}]dx$ admits no primitive. In fact,

$$H^n_{\mathrm{dR}}(\mathbf{G}_{m,k}/k) = \begin{cases} k & \text{n=0} \\ k\left[\frac{dx}{x}\right] & n = 1 \\ 0 & n \geq 2 \end{cases}$$

where $[dx/x]$ denotes the class of dx/x in de Rham cohomology.

Exercise 2.6. Compute the de Rham chomology of $\mathbf{A}^1_k \setminus \{p_1, \ldots, p_n\}$, where $p_i \in \mathbf{A}^1_k(k)$.

Example 2.7 (Punctured elliptic curve)**.** Let k be a field of characteristic 0 and $X = \operatorname{Spec} A$, where $A = k[x, y]/(y^2 - f(x))$, with $f = 4x^3 - g_2x - g_3$ satisfying $g_2^3 - 27g_3^2 \neq 0$. Recall that $\Omega^1_{X/k}$ is trivialized by

$$\omega = \frac{dx}{y} = 2\frac{dy}{f'(x)},$$

so that the de Rham complex of X over k is given by

$$0 \longrightarrow A \xrightarrow{d} A\omega \longrightarrow 0.$$

Let us compute d explicitly. Every element $h \in A$ can be written uniquely as $h = P + Qy$, with $P, Q \in k[x]$. Thus

$$dh = P'(x)dx + Q'(x)ydx + Q(x)dy = \left(\left(Q'f + \frac{1}{2}Qf'\right) + P'y\right)\omega.$$

If Q has leading term a_dx^d, then $Q'f + \frac{1}{2}Qf'$ has leading term $(4d+6)a_dx^{d+2}$ (note that $4d+6$ is never 0). Clearly, $dh = 0$ if and only if $Q = 0$ and $P' = 0$. Thus

$$H^0_{\mathrm{dR}}(X/k) = k.$$

To compute H^1, let $\eta = (R + Sy)\omega \in \Omega^1_{A/k}$, with $R, S \in k[x]$. By the above formula for dh, $Sy\omega$ is exact (take $Q = 0$ and P a primitive of S); thus we can write

$$\eta = R\omega + \text{exact}$$

Now, choosing appropriate Q (and $P = 0$), we can inductively kill the leading terms of R until we reach

$$\eta = (r_0 + r_1 x)\omega + \text{exact}.$$

We conclude that

$$H^1_{\mathrm{dR}}(X/k) = k[\omega] \oplus k[x\omega].$$

Exercise 2.8. Let $X : y^2 = f(x)$ be the affine part of a hyperelliptic curve of genus g over a field k of characteristic 0. Prove that $\dim_k H^1_{\mathrm{dR}}(X/k) = 2g$.

Exercise 2.9 (Base change)**.** Prove that if K is a field extension of k, then $H^n_{\mathrm{dR}}(X/k) \otimes_k K$ and $H^n_{\mathrm{dR}}(X \otimes_k K/K)$ are naturally isomorphic.

Remark 2.10 (Comparison theorem and periods)**.** Whenever k is of characteristic 0, the algebraic de Rham cohomology coincides with what we expect (say, from singular cohomology). This is no accident. Grothendieck proved that when $k = \mathbf{C}$, then the map

$$\mathrm{comp} : H^n_{\mathrm{dR}}(X/\mathbf{C}) \longrightarrow H^n(X(\mathbf{C}), \mathbf{C}) = \mathrm{Hom}(H_n(X(\mathbf{C}), \mathbf{Z}), \mathbf{C}), \qquad [\omega] \longmapsto \int_{\cdot} \omega$$

is an isomorphism. This is by no means trivial, since there are much less algebraic differental forms than analytic or smooth differential forms. We will come back to this in the next lecture.

For now, let us simply remark that the comparison isomorphism gives certain arithmetic invariants of a variety defined over $\mathbf{Q}$. Indeed, in this case there are two *different* $\mathbf{Q}$-structures on the cohomology of X: one given by $H^*_{\mathrm{dR}}(X/\mathbf{Q})$, and the other given by $H^*(X(\mathbf{C}), \mathbf{Q})$. What measures their difference are certain complex numbers called *periods* of X, given by

$$\langle \mathrm{comp}([\omega]), \sigma \rangle = \int_\sigma \omega, \qquad ([\omega] \in H^n_{\mathrm{dR}}(X/\mathbf{Q}),\ \sigma \in H_n(X(\mathbf{C}), \mathbf{Q})).$$

For instance, if $X = \mathbf{G}_{m,\mathbf{Q}}$ and $\sigma \in H_1(\mathbf{C}^\times, \mathbf{Z})$ is given by $t \longmapsto e^{2\pi i t}$, then

$$2\pi i = \int_\sigma \frac{dx}{x}$$

is a period.

3 Hypercohomology

For a projective scheme, it makes no sense to define the algebraic de Rham cohomology using global differential forms, since they may not exist (consider $\mathbf{P}^1$)! The general definition of algebraic de Rham cohomology involves the consideration of *hypercohomology*.

Recall that if $\Gamma : \mathcal{A} \longrightarrow \mathcal{B}$ is a left exact functor between abelian categories, and if $\mathcal{A}$ has enough injectives (i.e., objects $I \in \mathcal{A}$ for which $\mathrm{Hom}(-, I)$ is exact), then we define derived functors by

$$R^n\Gamma(A) := H^n(\Gamma(I^\bullet))$$

where $0 \longrightarrow A \longrightarrow I^\bullet$ is an injective resolution of $A \in \mathcal{A}$. Such definition does not depend on the chosen injective resolution.

In fact, we can use any acyclic resolution. Recall that $B \in \mathcal{A}$ is said to be *acyclic* for Γ if $R^n\Gamma(B) = 0$ for every $n \geq 1$. Then

$$R^n\Gamma(A) = H^n(\Gamma(B^\bullet))$$

where $0 \longrightarrow A \longrightarrow B^\bullet$ is an acyclic resolution of A (i.e., each B^i is acyclic).

Example 3.1 (Čech cohomology)**.** Let X be a separated scheme and consider the global sections functor $\Gamma(X, -) : \mathcal{A}b_X \longrightarrow \mathrm{Ab}$. Let $\mathcal{U} = (U_i)_i$ be an affine covering of X. Denote $U_{i_1\cdots i_r} = U_{i_1} \cap \cdots \cap U_{i_r}$ and let $j_{i_1\cdots i_r} : U_{i_1\cdots i_r} \hookrightarrow X$ be the inclusion. If $\mathcal{F}$ is a quasi-coherent abelian sheaf on X, then it follows from Serre's theorem that

$$\prod_{i_1,\ldots,i_r} j_{i_1\cdots i_r,*}\mathcal{F}|_{U_{i_1\cdots i_r}}$$

is acyclic. Thus

$$0 \longrightarrow \mathcal{F} \longrightarrow \prod_i j_{i,*}\mathcal{F}|_{U_i} \longrightarrow \prod_{i_1,i_2} j_{i_1 i_2,*}\mathcal{F}|_{U_{i_1 i_2}} \longrightarrow \cdots$$

is an acyclic resolution of $\mathcal{F}$. For instance, this proves that

$$H^1(X, \mathcal{F}) = \frac{\{(f_{ij})_{i,j} \in \prod_{i,j} \mathcal{F}(U_{ij}) \mid f_{ij} + f_{jk} = f_{ik} \text{ on } U_{ijk}\}}{\{\{(f_{ij})_{i,j} \in \prod_{i,j} \mathcal{F}(U_{ij}) \mid f_{ij} = f_i - f_j \text{ for some } (f_i)_i \in \prod_i \mathcal{F}(U_i)\}}$$

Finally, recall that whenever we have a short exact sequence

$$0 \longrightarrow A'' \longrightarrow A \longrightarrow A' \longrightarrow 0$$

in $\mathcal{A}$, we obtain a long exact sequence in cohomology

$$0 \longrightarrow R^0\Gamma(A'') \longrightarrow R^0\Gamma(A) \longrightarrow R^0\Gamma(A') \xrightarrow{\delta^0} R^1\Gamma(A'') \longrightarrow R^1\Gamma(A) \longrightarrow \cdots$$

where $\delta^i : R^i\Gamma(A') \longrightarrow R^{i+1}\Gamma(A'')$ are the *connecting morphisms.*

The idea of hypercohomology, or more generally of hyperderived functors, is to perform a similar formalism for cochain complexes $A^\bullet$ in $\mathcal{A}$ instead of objects. Let us denote by $C(\mathcal{A})$ the category of cochain complexes in $\mathcal{A}$.

Definition 3.2. A morphism $A^\bullet \longrightarrow B^\bullet$ in $C(\mathcal{A})$ is a *quasi-isomorphism* if it induces isomorphisms on the level of cohomology:

$$H^i(A^\bullet) \xrightarrow{\sim} H^i(B^\bullet)$$

for every i.

Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and $\Gamma : \mathcal{A} \longrightarrow \mathcal{B}$ be a left exact functor.

Definition 3.3. If $\mathcal{A}$ has enough injectives, and $A^\bullet \in C(\mathcal{A})$ is a left bounded complex, then we define

$$\mathbf{R}^n\Gamma(A^\bullet) \coloneqq H^n(\Gamma(B^\bullet))$$

where $A^\bullet \longrightarrow B^\bullet$ is a quasi-isomorphism with B^i acyclic for every i.

Such quasi-isomorphism always exists if $\mathcal{A}$ has enough injectives, and one may check that this definition does not depend on the chosen quasi-isomorphism.

Example 3.4. Consider the inclusion functor $\mathcal{A} \longrightarrow C(\mathcal{A})$ associating an object A of $\mathcal{A}$ to the complex concentrated in degree 0

$$0 \longrightarrow A^0 = A \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

A quasi-isomorphism $A \longrightarrow B^\bullet$ (with $B^i = 0$ for $i < 0$) is equivalent to a resolution $0 \longrightarrow A \longrightarrow B^\bullet$. Thus

$$\mathbf{R}^n\Gamma(A) = R^n\Gamma(A).$$

Exercise 3.5. For $A^\bullet \in C(\mathcal{A})$ and $m \in \mathbf{Z}$, we can consider the shifted complex $A[m]^\bullet = A^{\bullet+m}$. Prove that $\mathbf{R}^n\Gamma(A[m]^\bullet) = \mathbf{R}^{n+m}\Gamma(A^\bullet)$.

Exercise 3.6. Prove that for every short exact sequence of complexes

$$0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0$$

there exists a canonical "long exact sequence of cohomology"

$$0 \longrightarrow \mathbf{R}^0\Gamma(A^\bullet) \longrightarrow \mathbf{R}^0\Gamma(B^\bullet) \longrightarrow \mathbf{R}^0\Gamma(C^\bullet) \xrightarrow{\delta^0} \mathbf{R}^1\Gamma(A^\bullet) \longrightarrow \mathbf{R}^1\Gamma(B^\bullet) \longrightarrow \cdots$$

Recall that a spectral sequence is a family of complexes ("page r")

$$(E_r^{p,q}, d_r), \qquad d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$$

such that

$$E_{r+1}^{p,q} = \frac{\ker(d_r : E_r^{p,q} \longrightarrow E^{p+r,q-r+1})}{\operatorname{im}(d_r : E^{p-r,q+r-1} \longrightarrow E_r^{p,q})}.$$

Here $p, q \geq 0$ and $r \geq 1$ (or $r \geq 2$).

For instance, the first page of a spectral sequence looks as follows:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
E_1^{0,2} & \longrightarrow & E_1^{1,2} & \longrightarrow & E_1^{2,2} & \longrightarrow & \cdots \\
E_1^{0,1} & \longrightarrow & E_1^{1,1} & \longrightarrow & E_1^{2,1} & \longrightarrow & \cdots \\
E_1^{0,0} & \longrightarrow & E_1^{1,0} & \longrightarrow & E_1^{2,0} & \longrightarrow & \cdots
\end{array}$$

The second page is obtained by taking cohomology groups of the complexes in the first page, and so on.

Theorem 3.7. *Let $\mathcal{A}$, $\mathcal{B}$ and Γ be as before. Suppose that $A^\bullet$ is a left bounded complex having a finite filtration*

$$A^\bullet = F^0A^\bullet \supset F^1A^\bullet \supset F^2A^\bullet \supset \cdots \supset F^NA^\bullet \supset 0$$

Then there exists a unique spectral sequence $(E_r^{p,q}, d_r)_{r\geq 1}$ such that:

1. *$E_1^{p,q} = \mathbf{R}^{p+q}\Gamma(F^pA^\bullet/F^{p+1}A^\bullet)$ and $d_1 : E_1^{p,q} \longrightarrow E_1^{p+1,q}$ is given by the connecting morphism δ^{p+q} of the long exact sequence associated to*

$$0 \longrightarrow F^{p+1}A^\bullet/F^{p+2}A^\bullet \longrightarrow F^pA^\bullet/F^{p+2}A^\bullet \longrightarrow F^pA^\bullet/F^{p+1}A^\bullet \longrightarrow 0$$

2. If we denote

$$F^i\mathbf{R}^n\Gamma(A^\bullet) \coloneqq \operatorname{im}(\mathbf{R}^n\Gamma(F^iA^\bullet) \longrightarrow \mathbf{R}^n\Gamma(A^\bullet))$$

then

$$E_r^{p,q} = F^p\mathbf{R}^{p+q}\Gamma(A^\bullet)/F^{p+1}\mathbf{R}^{p+q}\Gamma(A^\bullet).$$

for $r \gg 0$.