Calculus on Schemes - Lecture 7

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1 Algebraic de Rham cohomology 2

We now define the de Rham cohomology of an arbitraty morphism of schemes.

Definition 1.1. Let $\pi : X \longrightarrow S$ be a morphism of schemes. For every $n \in \mathbf{N}$, the nth (algebraic) *de Rham cohomology* of X over S is defined by

$$\mathcal{H}^n_{\mathrm{dR}}(X/S) \coloneqq \mathbf{R}^n\pi_*(\Omega^\bullet_{X/S})$$

Note that $\mathcal{H}^n_{\mathrm{dR}}(X/S)$ is a *sheaf* of $\mathcal{O}_S$-modules on S. Let us denote

$$H^n_{\mathrm{dR}}(X/S) = \Gamma(S, \mathcal{H}^n_{\mathrm{dR}}(X/S))$$

So that, if $S = \operatorname{Spec} R$ is affine, then $H^1_{\mathrm{dR}}(X/S) =: H^1_{\mathrm{dR}}(X/R)$ is an R-module.

Remark 1.2 (Cohomology with coefficients)**.** To every module with connection $(\mathcal{E}, \nabla)$ on X, we can define unique morphisms of abelian sheaves

$$\nabla^i : \mathcal{E} \otimes \Omega^i_{X/S} \longrightarrow \mathcal{E} \otimes \Omega^{i+1}_{X/S}$$

satisfying $\nabla^i(e \otimes \omega) = e \otimes d\omega + (-1)^i\nabla(e) \wedge \omega$. The connection $\nabla = \nabla^0$ is integrable if and only if (exercise!)

$$\mathcal{E} \otimes \Omega^\bullet_{X/S} : \qquad 0 \longrightarrow \mathcal{E} \xrightarrow{\nabla^0} \mathcal{E} \otimes \Omega^1_{X/S} \xrightarrow{\nabla^1} \mathcal{E} \otimes \Omega^2_{X/S} \longrightarrow \cdots$$

is a complex. In this case, we define

$$\mathcal{H}^n_{\mathrm{dR}}(X/S, (\mathcal{E}, \nabla)) \coloneqq \mathbf{R}^n\pi_*(\mathcal{E} \otimes \Omega^\bullet_{X/S}).$$

Example 1.3. Suppose that S is separated and that $\pi : X \longrightarrow S$ is affine. Then $\Omega^i_{X/S}$ is acyclic for π_* for every i (Serre's theorem), and we get

$$\mathcal{H}^n_{\mathrm{dR}}(X/S) = H^n(\pi_*\Omega^\bullet_{X/S}).$$

In particular, when $S = \operatorname{Spec} R$, we have

$$H^n_{\mathrm{dR}}(X/R) = H^n(\Gamma(X, \Omega^\bullet_{X/R}))$$

and we recover our original definition of the de Rham cohomology for smooth affine schemes over a field.

Let us assume that $\Omega^\bullet_{X/S}$ is bounded and consider the *stupid filtration* $(\sigma_{\geq p}\Omega^\bullet_{X/S})_{p\geq 0}$ on $\Omega^\bullet_{X/S}$. Here, $\sigma_{\geq p}\Omega^\bullet_{X/S}$ is the subcomplex of $\Omega^\bullet_{X/S}$ obtained by a truncation at degree p:

$$\sigma_{\geq p}\Omega^\bullet_{X/S}: \qquad 0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \Omega^p_{X/S} \longrightarrow \Omega^{p+1}_{X/S} \longrightarrow \cdots$$

This is a finite filtration with graded pieces

$$\sigma_{\geq p}\Omega^\bullet_{X/S}/\sigma_{\geq p+1}\Omega^\bullet_{X/S} = \Omega^p_{X/S}[-p]$$

Thus it gives rise to a spectral sequence, so called *Hodge to de Rham spectral sequence*[1],

$$E_1^{p,q} = R^q\pi_*\Omega^p_{X/S} \Rightarrow \mathcal{H}^{p+q}_{\mathrm{dR}}(X/S).$$

The induced filtration on $\mathcal{H}^n_{\mathrm{dR}}(X/S)$ is called the *Hodge filtration*:

$$F^i\mathcal{H}^n_{\mathrm{dR}}(X/S) = \mathrm{im}(\mathbf{R}^n\pi_*(\sigma_{\geq i}\Omega^\bullet_{X/S}) \longrightarrow \mathbf{R}^n\pi_*(\Omega^\bullet_{X/S})).$$

Example 1.4 (Curves)**.** Let k be a field and X be a smooth, projective, and geometrically connected curve over k. The first page of the Hodge to de Rham spectral sequence is:

$$H^1(X,\mathcal{O}_X) \xrightarrow{H^1(d)} H^1(X,\Omega^1_{X/k})$$

$$H^0(X,\mathcal{O}_X) \xrightarrow{H^0(d)} H^0(X,\Omega^1_{X/k})$$

where $d : \mathcal{O}_X \longrightarrow \Omega^1_{X/k}$ is the differential. Since $H^0(X,\mathcal{O}_X) = k$, we have $H^0(d) = 0$. Now, we have a commutative diagram

$$\begin{array}{ccc} H^1(X,\mathcal{O}_X) & \xrightarrow{H^1(d)} & H^1(X,\Omega^1_{X/k}) \\ \sim\downarrow & & \downarrow\sim \\ H^0(X,\Omega^1_{X/k})^\vee & \xrightarrow[H^0(d)^\vee]{} & H^0(X,\mathcal{O}_X)^\vee \end{array}$$

where the vertical isomorphisms are given by Serre duality. In particular, $H^1(d) = 0$. This proves that the Hodge to de Rham spectral sequence degenerates at page 1. It follows that:

1. $H^0_{\mathrm{dR}}(X/k) \cong H^0(X,\mathcal{O}_X) = k$
2. $F^1H^1_{\mathrm{dR}}(X/k) \cong H^0(X,\Omega^1_{X/k})$ and $H^1_{\mathrm{dR}}(X/k)/F^1H^1_{\mathrm{dR}}(X/k) \cong H^1(X,\mathcal{O}_X)$.
3. $H^2_{\mathrm{dR}}(X/k) = F^1H^2_{\mathrm{dR}}(X/k) \cong H^1(X,\Omega^1_{X/k}) \cong k$

In particular $\dim H^1_{\mathrm{dR}}(X/k) = 2g$, where g is the genus of X.

Remark 1.5. One can also obtain the exact sequence

$$0 \longrightarrow H^0(X,\Omega^1_{X/k}) \longrightarrow H^1_{\mathrm{dR}}(X/k) \longrightarrow H^1(X,\mathcal{O}_X) \longrightarrow 0$$

from the long exact sequence in cohomology associated to the short exact sequence of complexes

$$0 \longrightarrow \Omega^\bullet_{X/k}[-1] \longrightarrow \Omega^\bullet_{X/k} \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

[1] It follows from an exercise from last lecture that $\mathbf{R}^{p+q}\pi_*\Omega^p_{X/S}[-p] = R^q\pi_*\Omega^p_{X/S}$.

Exercise 1.6. Prove analogous results in the relative situation, i.e., when $X \longrightarrow S$ is a projective smooth morphisms of relative dimension 1.

Exercise 1.7. Prove that $H^{\mathrm{odd}}_{\mathrm{dR}}(\mathbf{P}^n_k/k) = 0$ and $H^{\mathrm{even}}_{\mathrm{dR}}(\mathbf{P}^n_k/k) \cong k$.

Remark 1.8. In general, when X is smooth and projective over $\mathbf{C}$, then it follows from Hodge theory and GAGA that

$$H^q(X, \Omega^p_{X/\mathbf{C}}) \Rightarrow H^{p+q}_{\mathrm{dR}}(X/\mathbf{C})$$

degenerates at page 1. In particular, if $(F^i)_i$ denotes the Hodge filtration on $H^n_{\mathrm{dR}}(X/\mathbf{C})$, we get canonical isomorphisms

$$F^i/F^{i+1} \cong H^{n-i}(X, \Omega^i_{X/\mathbf{C}}).$$

It is also possible to compute de Rham cohomology via Čech complexes:

Example 1.9. Suppose that X is a separated scheme of finite type over k and consider an affine covering $X = \bigcup_i U_i$. If $j_{i_1\cdots i_p} : U_{i_1\cdots i_p} \longrightarrow X$ denotes the inclusion, then we have a double complex

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \prod_{i_1,i_2} j_{i_1i_2,*}\mathcal{O}_{U_{i_1i_2}} & \longrightarrow & \prod_{i_1,i_2} j_{i_1i_2,*}\Omega^1_{U_{i_1i_2}/k} & \longrightarrow & \prod_{i_1,i_2} j_{i_1i_2,*}\Omega^2_{U_{i_1i_2}/k} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \prod_i j_{i,*}\mathcal{O}_{U_i} & \longrightarrow & \prod_i j_{i,*}\Omega^1_{U_i/k} & \longrightarrow & \prod_i j_{i,*}\Omega^2_{U_i/k} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \Omega^1_{X/k} & \longrightarrow & \Omega^2_{X/k} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Since the ith vertical complex gives a resolution of $\Omega^i_{X/k}$, we obtain a quasi-isomorphism $\Omega^\bullet_{X/k} \longrightarrow \mathrm{Tot}^\bullet$, were $\mathrm{Tot}^\bullet$ denotes the total complex of the double Čech complex. Now, using that Tot^i is acyclic for every i, we can obtain fairly explicit descriptions of the de Rham cohomology groups. For instance,

$$H^1_{\mathrm{dR}}(X/k) = \frac{\left\{((f_{ij})_{i,j}, (\omega_i)_i) \in \prod_{i,j} \mathcal{O}_X(U_{ij}) \oplus \prod_i \Omega^1_{X/k}(U_i) \;\middle|\; \omega_i - \omega_j = df_{ij} \text{ over } U_{ij}\right\}}{\left\{((f_i - f_j)_{ij}, (df_i)_i) \in \prod_{i,j} \mathcal{O}_X(U_{ij}) \oplus \prod_i \Omega^1_{X/k}(U_i) \;\middle|\; \text{for some } (f_i)_i \in \prod_i \mathcal{O}_X(U_i)\right\}}$$

Finally, let us briefly mention that algebraic de Rham cohomology has an obvious functoriality. In fact, if $\varphi : X \longrightarrow Y$ is a morphism of S-schemes, then the canonical morphism $\varphi^*\Omega^1_{Y/S} \longrightarrow \Omega^1_{X/S}$ of $\mathcal{O}_X$-modules induces a morphism of complexes $\varphi^*\Omega^\bullet_{Y/S} \longrightarrow \Omega^\bullet_{X/S}$, which yields a natural morphism of $\mathcal{O}_S$-modules

$$\varphi^* : \mathcal{H}^n_{\mathrm{dR}}(Y/S) \longrightarrow \mathcal{H}^n_{\mathrm{dR}}(X/S).$$

Exercise 1.10. Let X be an elliptic curve over a field k of characteristic 0 and $U = X \setminus \{\infty\}$. Prove that the inclusion $j : U \longrightarrow X$ induces an isomorphism $j^* : H^1_{\mathrm{dR}}(X/k) \xrightarrow{\sim} H^1_{\mathrm{dR}}(U/k)$.

2 The Gauss-Manin connection

For simplicity, we fix a field k of characteristic 0, and we work with smooth k-schemes of finite type X and S.

Let $\pi : X \longrightarrow S$ be a smooth morphism. We will show that, for every $n \geq 0$, there exists a canonical integrable connection

$$\nabla : \mathcal{H}^n_{\mathrm{dR}}(X/S) \longrightarrow \mathcal{H}^n_{\mathrm{dR}}(X/S) \otimes \Omega^1_{S/k}.$$

We follow the approach of Katz-Oda (see [1]).

Let us denote for simplicity $\Omega^*_{X/k} = \Omega^*_X$, $\Omega^*_{S/k} = \Omega^*_S$, and $\mathcal{H}^*_{\mathrm{dR}}(X/S) = \mathcal{H}^*$. Then we can consider the following finite filtration on $\Omega^\bullet_X$:

$$F^i = \mathrm{im}(\pi^*\Omega^i \otimes \Omega^{\bullet - i} \longrightarrow \Omega^\bullet_X)$$

where the above map is given by the wedge product. Note that $F^i = 0$ for i greater then the relative dimension of X over S.

Lemma 2.1. *We have*

$$F^i/F^{i+1} = \pi^*\Omega^i_S \otimes \Omega^{\bullet - i}_{X/S}$$

Proof. Since π is smooth, the sequence $0 \longrightarrow \pi^*\Omega^1_S \longrightarrow \Omega^1_X \longrightarrow \Omega^1_{X/S} \longrightarrow 0$ is exact and locally split. Thus we have an isomorphism

$$\bigoplus_i \pi^*\Omega^i_S \otimes \Omega^{p-i}_{X/S} \xrightarrow{\sim} \bigwedge^p \Omega^1_X = \Omega^p_X$$

and we get

$$F^{i,p} = \bigoplus_{j \geq i} \pi^*\Omega^j_S \otimes \Omega^{p-j}_{X/S}$$

The assertion easily follows. ■

Let $E^{p,q}_1 \Rightarrow E^{p+q}$ be the spectral sequence associated to the finite filtration $(F^i)_i$ and the functor π_*.

Lemma 2.2. *We have* $E^{p,q}_1 = \mathcal{H}^q_{\mathrm{dR}}(X/S) \otimes \Omega^p_S$.

Proof. Since Ω^1_X is locally free, it follows from last lemma and the "projection formula" that

$$E^{p,q}_1 = \mathbf{R}^{p+q}\pi_*(F^{p,\bullet}/F^{p+1,\bullet}) = \mathbf{R}^{p+q}\pi_*(\pi^*\Omega^p_S \otimes \Omega^{\bullet - p}_X) = \Omega^p_S \otimes \mathbf{R}^q\pi_*(\Omega^\bullet_X).$$

■

Here is how the first page of this spectral sequence looks like:

$$\mathcal{H}^2 \longrightarrow \mathcal{H}^2 \otimes \Omega^1_S \longrightarrow \mathcal{H}^2 \otimes \Omega^2_S \longrightarrow \cdots$$

$$\mathcal{H}^1 \longrightarrow \mathcal{H}^1 \otimes \Omega^1_S \longrightarrow \mathcal{H}^1 \otimes \Omega^2_S \longrightarrow \cdots$$

$$\mathcal{H}^0 \longrightarrow \mathcal{H}^0 \otimes \Omega^1_S \longrightarrow \mathcal{H}^0 \otimes \Omega^2_S \longrightarrow \cdots$$

Note that $\mathcal{H}^0$ is an $\mathcal{O}_S$-algebra, and that the above maps $\mathcal{H}^0 \otimes \Omega^i_S \longrightarrow \mathcal{H}^0 \otimes \Omega^{i+1}_S$ are given by $\mathrm{id} \otimes d$, where $d : \Omega^i_S \longrightarrow \Omega^{i+1}_S$ is the usual differential.

Proposition 2.3. *The wedge product of differential forms induces a product structure on the spectral sequence $E_1^{p,q} \Rightarrow E^{p+q}$, that is, a family of bilinear maps*

$$E_r^{p,q} \times E_r^{p',q'} \longrightarrow E_r^{p+p',q+q'}, \qquad (e,e') \longmapsto e \cdot e'$$

such that

1. $e \cdot e' = (-1)^{(p+q)(p'+q')} e' \cdot e$
2. $d_r(e \cdot e') = d_r(e) \cdot e' + (-1)^{p+q} e \cdot d_r(e')$.

Proof. The proof is formal and we leave it as an exercise. Note that if ω (resp. η) is a section of F^i (resp. F^j), then $\omega \wedge \eta$ is a section of F^{i+j}. ■

Corollary 2.4. *The differential in page 1*

$$d_1 : \mathcal{H}^n \longrightarrow \mathcal{H}^n \otimes \Omega_S^1$$

is an integrable connection.

Proof. That d_1 is a connection follows from property 2 above and from the fact that $\mathcal{O}_S$ injects into $\mathcal{H}^0$. It is automatically integrable since we know a priori that $\mathcal{H}^n \otimes \Omega_S^\bullet$ is a complex! ■

This is the Gauss-Manin connection: $\nabla = d_1$. Let us now collect some corollaries of the existence of such connection.

Lemma 2.5. *Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$, and suppose that A contains its residue field k. Suppose that for every $a \in \mathfrak{m} \setminus \{0\}$, there exists a derivation $D \in \mathrm{Der}_k(A)$ such that $v_\mathfrak{m}(Da) < v_\mathfrak{m}(a)$, where $v_\mathfrak{m}(x) := \max\{i \mid x \in \mathfrak{m}^i\}$. Then every finitely generated A-module with a k-connection is free.*

Proof. Exercise. Hint: if E is a finitely generated A-module with a k-connection, take $e_1, \ldots, e_r \in E$ reducing to a basis of $E \otimes_A k$, and show that they are A-linearly independent. ■

Corollary 2.6. *Suppose that π is proper. Then the coherent $\mathcal{O}_S$-module $\mathcal{H}^n$ is a vector bundle over S.*

Proof. It is sufficient to prove that $\mathcal{H}^n$ is locally free. Since S is smooth over k, this follows from the above lemma. Indeed, by flat descent, we can assume $k = \overline{k}$. Let $(s_1, \ldots, s_m)$ be local coordinates at a closed point $p \in S$. Then we can lift the derivations $\partial/\partial s_i$ to the completion $\hat{\mathcal{O}}_{S,p} \cong k[\![s_1, \ldots, s_m]\!]$, and now the hypotheses of the above lemma are easy to check. ■

From now on, we keep the hypothesis that π is proper.

Corollary 2.7. *Given a Cartesian square of k-schemes*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & \square & \downarrow \\ S' & \xrightarrow[\varphi]{} & S \end{array}$$

where S' is also smooth and of finite type over k, there is a canonical isomorphism of $\mathcal{O}_{S'}$-modules

$$\varphi^* \mathcal{H}^n_{\mathrm{dR}}(X/S) \xrightarrow{\sim} \mathcal{H}^n_{\mathrm{dR}}(X'/S').$$

In particular, for every closed point $p \in S$, we have a canonical isomorphism

$$\mathcal{H}^n_{\mathrm{dR}}(X/S)(p) \cong H^n_{\mathrm{dR}}(X_p/k_p).$$

Proof. Since $\pi : X \longrightarrow S$ is proper and the de Rham cohomology sheaves are locally free, this follows from the usual "cohomology and base change" techniques. See [2] Section 8. ■

There is also a concept of pullback for connections. If $(\mathcal{E}, \nabla)$ is a vector bundle with k-connection on S, and $\varphi : S' \longrightarrow S$ is a morphism of k-schemes, then the pullback connection $\varphi^*\nabla$ on the vector bundle $\varphi^*\mathcal{E}$ over S' is the unique k-connection such that

$$(\varphi^*\nabla)(\varphi^* e) = \varphi^*(\nabla e) \tag{2.1}$$

for every local section e of $\mathcal{E}$.

We next state a naturality statement for the Gauss-Manin connection, the proof of which we leave as an exercise: it easily follows from last corollary and from the explicit construction of the Gauss-Manin connection as a differential in a spectral sequence.

Proposition 2.8. *Given a Cartesian square of k-schemes*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & \square & \downarrow \\ S' & \xrightarrow[\varphi]{} & S \end{array}$$

where S' is also smooth and of finite type over k, if $\nabla : \mathcal{H}^n_{\mathrm{dR}}(X/S) \longrightarrow \mathcal{H}^n_{\mathrm{dR}}(X/S) \otimes \Omega^1_{S/k}$ denotes the Gauss-Manin connection, then $\varphi^*\nabla$ is the Gauss-Manin connection on $\mathcal{H}^n_{\mathrm{dR}}(X'/S')$ after the identification $\mathcal{H}^n_{\mathrm{dR}}(X'/S') \cong \varphi^*\mathcal{H}^n_{\mathrm{dR}}(X/S)$ of last corollary.

There is also a similar result concerning a base change on the base field. Let k' be a field containing k, and consider a diagram of schemes

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & \square & \downarrow \\ S' & \longrightarrow & S \\ \downarrow & \square & \downarrow \\ \operatorname{Spec} k' & \longrightarrow & \operatorname{Spec} k \end{array}$$

where both squares are Cartesian. If $(\mathcal{E}, \nabla)$ is a vector bundle with a k-connection on S, then there exists a unique k'-connection ∇' on the vector bundle $\mathcal{E}' = \mathcal{E} \otimes_k k'$ over S' satisfying the same property of (2.1). In particular, if $\mathcal{E} = \mathcal{H}^n_{\mathrm{dR}}(X/S)$ and ∇ is the Gauss-Manin connection, then we get a k'-connection

$$\nabla' : \mathcal{H}^n_{\mathrm{dR}}(X'/S') \longrightarrow \mathcal{H}^n_{\mathrm{dR}}(X'/S') \otimes \Omega^1_{S'/k'}$$

It is not difficult to show, again from the explicit constructions, that ∇' is in fact the Gauss-Manin connection on $\mathcal{H}^n_{\mathrm{dR}}(X'/S')$.

Finally, let us state a comparison theorem which follows immediately from GAGA and from the "holomorphic Poincaré lemma".

Theorem 2.9. *Let $k = \mathbf{C}$. Then there exists a canonical isomorphism of vector bundles with connection on S^{an}*

$$(\mathcal{H}^n_{\mathrm{dR}}(X/S)^{\mathrm{an}}, \nabla^{\mathrm{an}}) \cong (R^n\pi^{\mathrm{an}}_* \mathbf{C}_{X^{\mathrm{an}}} \otimes_{\mathbf{C}} \mathcal{O}_{S^{\mathrm{an}}}, \mathrm{id} \otimes d).$$

In particular, if α is a section of $\mathcal{H}^n_{\mathrm{dR}}(X/S)^{\mathrm{an}}$, and σ is a section of $R_n\pi^{\mathrm{an}}_* \mathbf{Z}_{X^{\mathrm{an}}}$, we have

$$d\left(\int_\sigma \alpha\right) = \int_\sigma \nabla\alpha.$$

The above equation can be used to compute the Gauss-Manin connection explicitly.

References

[1] Katz, N. M., Oda, T., *On the differentiation of De Rham cohomology classes with respect to parameters*. J. Math. Kyoto Univ. 8-2, 199-213, 1968.

[2] Katz, N. M., *Nilpotent connections and the monodromy theorem : applications of a result of Turritin*. Publications mathématiques de l'I.H.E.S., tome 39, p. 175-232, 1970.