Calculus on Schemes - Lecture 8 18/06/19

Last time:

- Alg. de Rham cohom $\begin{array}{c} X \\ \pi \downarrow \\ S \end{array}$ $\rightsquigarrow$ $\mathcal{H}^n_{dR}(X/S) := \mathbb{R}^n \pi_*(\Omega^\bullet_{X/S})$

  Hodge to de Rham spec seq $R^q \pi_* \Omega^p_{X/S} \Rightarrow \mathcal{H}^{p+q}_{dR}(X/S)$

- Gauß-Manin connection $\nabla : \mathcal{H}^n_{dR} \longrightarrow \mathcal{H}^n_{dR} \otimes \Omega^1_{S/k}$

  $\begin{array}{c} X \\ \downarrow \text{sm} \\ S \\ \downarrow \text{sm} \\ \operatorname{Spec} k \end{array}$ $\Rightarrow$ $\mathcal{H}^n_{dR}$ v. bun

  compatibility w/ base change,

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1) Picard-Fuchs equations and regular singularities

Ex (Legendre family)

$$\begin{array}{ccc} E & & E_\lambda : y^2 = x(x-1)(x-\lambda) \\ \pi \downarrow & & | \\ S = \mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,\infty\} \ni \lambda & & \end{array}$$

Can trivialise $\mathcal{H}^1_{dR}(E/S)$ by (ω, η) where

$(\omega_\lambda, \eta_\lambda) = \left(\frac{dx}{2y}, x\frac{dx}{2y}\right)$ basis of $H^1_{dR}(E_\lambda/\mathbb{Q}(\lambda))$

Then (exercise):

GM conn.

$$\nabla_{\frac{d}{d\lambda}} (\omega \quad \eta) = (\omega \quad \eta) \begin{pmatrix} \frac{1}{2(1-\lambda)} & \frac{1}{2(1-\lambda)} \\ -\frac{1}{2\lambda(1-\lambda)} & -\frac{1}{2(1-\lambda)} \end{pmatrix}$$

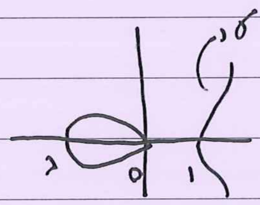
Picard-Fuchs eqn: write $\nabla_{\frac{d}{d\lambda}}^2 \omega$ in $(\omega, \nabla_{\frac{d}{d\lambda}} \omega)$:

$$\nabla_{\frac{d}{d\lambda}} \left(\nabla_{\frac{d}{d\lambda}} \omega \right) = \frac{2\lambda - 1}{\lambda(1-\lambda)} \nabla_{\frac{d}{d\lambda}} \omega + \frac{1}{4\lambda(1-\lambda)} \omega$$

If γ is a section of $R_1 \pi_*^{an} \mathbb{Q}_{E^{an}}$, then

$\varpi := \int_\gamma \omega$ satisfies $\frac{d\varpi}{d\lambda} = \int_\gamma \nabla_{\frac{d}{d\lambda}} \omega$, so

$$\lambda(1-\lambda) \frac{d^2\varpi}{d\lambda^2} + (1-2\lambda)\frac{d\varpi}{d\lambda} - \frac{1}{4} = 0$$

e.g. 

$$\varpi = \int_1^\infty \frac{dx}{\sqrt{x(x-1)(x-\lambda)}} = \pi \sum_{n=0}^{\infty} \binom{-1/2}{n}^2 \lambda^n$$

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In general, S/k smooth curve, K = function field

$(\mathcal{E}, \nabla)$ vb w/ conn. on S of rk r

$\rightsquigarrow$ $(\mathcal{E} \otimes K, \nabla)$ vb w/ conn on Spec K

Fix $\theta \in T_{S/k} \otimes K$.

<u>Def</u> $e \in \mathcal{E} \otimes K$ is <u>cyclic</u> if $(e, \nabla_\theta e, \dots, \nabla_\theta^{r-1} e)$

$\Rightarrow$ generates $\mathcal{E} \otimes K$

$\rightsquigarrow$ get eqn. $\nabla_\theta^r e + f_1 \nabla_\theta^{r-1} e + \cdots + f_r e = 0$, $f_i \in K$

(E): $\theta^r + f_1 \theta^{r-1} + \cdots + f_r = 0$ is the ass. homogeneous liner eq.

When $\nabla = GM$ $\rightsquigarrow$ Picard-Fuchs.

Suppose: $k = \mathbb{C}$, $S \subset \mathbb{P}^1_{\mathbb{C}} \setminus \{\infty\}$, take $\theta = \frac{d}{dz}$.

$e \in \Gamma(S, \mathcal{E}) \Rightarrow f_i$ regular at S

$p \in \mathbb{P}^1 \setminus S$ are the singular pts of (E)

Def $p \in \mathbb{A}^1 \setminus S$ is a regular singularity if

$\forall$ D $\forall$ hol. sol. u on D, $\exists$ $N \geq 0$ st

$$|u(z)| = O(|z-p|^{-N}), \quad z \to p$$

and similarly for $p = \infty$ (after $z \mapsto \frac{1}{z}$)

Good for Frobenius method

Prop. (Fuchs)

$p \in \mathbb{A}^1 \setminus S$ is a regular singularity $\Leftrightarrow$ $(z-p)^i f_i(z)$ is hol. at $z = p$ $\forall i = 1, \ldots, r$. Similarly for $p = \infty$.

[Actually: $f_i = \dfrac{g_i}{\prod_j (z-p_j)^i}$, $\deg g_i \leq i(r-1)$ $(\mathbb{A}^1 \setminus S = \{p_j\})$]

Remark on Hilbert's XXI problem? "Riemann-Hilbert problem"

Ex $\frac{d^2}{dz^2} + \frac{(1-2z)}{z(1-z)}\frac{d}{dz} - \frac{1}{4z(1-z)} = 0$

$\Rightarrow$ $0, 1, \infty$ are all regular singularities.

2) Regularity of the GM connection

$X/_k$ smooth, connected, f.t. $\overset{\dim n}{V}$, $\mathrm{char}(k) = 0$

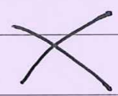
Def $D \hookrightarrow X$ is a normal crossings divisor

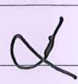
if $\exists$ étale covering $(U_i \to X)_i$ st

$$\begin{array}{ccc} D \times_X U_i & \longrightarrow & U_i \\ \downarrow & \square & \downarrow \text{ét} \\ V(x_1 \cdots x_r) & \longrightarrow & \mathbb{A}^n_k \end{array}$$

If Zariski-covering $\rightsquigarrow$ simple normal crossings

Ex 1) $D = D_1 \cup \cdots \cup D_r$, $r \leq n$, D_i smooth

"transversal intersections" $\Rightarrow$ D SNCD 

2) $D = V(y^2 - x^2(x+1)) \hookrightarrow \mathbb{A}^2$ NCD but not SNCD 

Thm (Hironaka)

later

- $\exists$ $\bar{X}/_k$ smooth, proper st $\bar{X} \setminus X = D$ NCD

- Given $X \to S$, $\exists$

$$\begin{array}{ccccc} E & \hookrightarrow & \bar{X} & \hookleftarrow & X \\ \downarrow & \square & \downarrow & \square & \downarrow \\ D & \hookrightarrow & \bar{S} & \hookleftarrow & S \end{array}$$

E, D NCD

(Can assume SNCD!)

Let $D \hookrightarrow X$ NCD, $j : U = X \setminus D \hookrightarrow X$

$(\rightsquigarrow \Omega^p_{X/k} \hookrightarrow j_* \Omega^p_{U/k})$

<u>Def</u> $\Omega^p_{X/k}(\log D)$ = subsheaf of $j_* \Omega^p_{U/k}$ of

$w^{\in j_* \Omega^p_{U/k}}$ st w and dw have "at most a simple pole on D".

[f loc. eqn. of D $\Rightarrow$ fw and $f\,dw$ ~~extends to~~ "extends to X"]

<u>Ex</u> $\Omega^1_{\mathbb{A}^1/k}(\log 0) = \mathcal{O}_{\mathbb{A}^1} \frac{dx}{x}$

<u>Prop</u>. 1) $\Omega^p_{X/k}(\log D)$ is a v. bun. $\forall p$

2) $\Omega^p_{X/k}(\log D) = \Lambda^p \Omega^1_{X/k}(\log D)$

3) $d : \Omega^p_{X/k} \to \Omega^{p+1}_{X/k}$ extends to $d : \Omega^p_{X/k}(\log D) \to \Omega^{p+1}_{X/k}(\log D)$

4) $U \xrightarrow{et} X$ st $D \times_X U = V(x_1, \cdots x_r)$ $\Rightarrow$ $\Omega^1_{U/k}(\log D_U)$

$= \langle \frac{dx_1}{x_1}, \ldots, \frac{dx_r}{x_r}, dx_{r+1}, \ldots, dx_n \rangle$.

Proof. Exercise. ∎

Hironaka!

<u>Def</u> Let $(\mathcal{E}, \nabla)$ be a v. bun. w/ connection on

X/k. ∇ is <u>regular</u> if $\forall$ "Hironaka compactification"

$\bar{X} = X \cup D$, $\exists\ \bar{\mathcal{E}}$ v. bun. $/\bar{X}$, extending $\mathcal{E}$, $\bar{\nabla} : \bar{\mathcal{E}} \to \bar{\mathcal{E}} \otimes \Omega^1_{\bar{X}/k}(\log D)$ extending ∇.

Ex $(\mathcal{O}_X, d)$ is always regular

Ex X curve, $p \in \bar{X} \setminus X$, x coordinate at p, $\mathcal{E} \simeq \mathcal{O}_X^{\oplus r}$ in a neighb. of p

$\Rightarrow \nabla = d + \Omega$, $\Omega = (w_{ij})$. ∇ is "regular at p" $\Leftrightarrow$ $w_{ij} = f_{ij} \frac{dx}{x}$, $f_{ij} \in \mathcal{O}_{X,p}$.

Exercise Prove that a ODE is Fuchsian $\Leftrightarrow$ corresponding connection is regular!

Griffiths, Deligne, Katz

Thm $X, S/k$ smooth connected, $\pi: X \to S$ proper smooth
↳ char 0

$\forall n$, $\nabla: \mathcal{H}^n_{dR}(X/S) \to \mathcal{H}^n_{dR}(X/S) \otimes \Omega^1_{S/k}$ is regular.

"Proof": • "∇ regular $\Leftrightarrow$ $\varphi^* \nabla$ regular $\forall$ $C \to S$, C smooth curve". GM commutes w/ base change

$\Rightarrow$ can assume S is a curve

• Consider (Hironaka)

$$\begin{array}{ccccc} X & \hookrightarrow & \bar{X} & \hookleftarrow & E \\ \downarrow & & \downarrow \pi & & \downarrow \\ S & \hookrightarrow & \bar{S} & \hookleftarrow & D \end{array}$$

Set $\Omega^1_{\bar{X}/\bar{S}}(\log E/D) := \Omega^1_{\bar{X}/k}(\log E) / \pi^* \Omega^1_{\bar{S}/k}(\log D)$

$\rightsquigarrow$ $0 \to \pi^* \Omega^1_{\bar{S}}(\log D) \to \Omega^1_{\bar{X}}(\log E) \to \Omega^1_{\bar{X}/\bar{S}}(\log E/D) \to 0$

exact and loc. split

- Set $\Omega^p_{\bar{X}/\bar{S}}(\log E/\log D)$ and apply Katz-Oda formalism to $(\Omega^\bullet(\log E/\log D), d)$

$\leadsto$ get coherent $\bar{\mathcal{H}}^n$ with $\bar{\nabla} : \bar{\mathcal{H}}^n \to \bar{\mathcal{H}}^n \otimes \Omega^1_{\bar{S}}(\log D)$

st $(\bar{\mathcal{H}}^n, \bar{\nabla})|_S = (\mathcal{H}, \nabla)$.

- $\bar{\mathcal{H}}^n$ is a vb: ~~no tors.~~ torsion-free outside D

$S \subset \bar{S}$ dense $\Rightarrow$ torsion-free everywhere

$\Rightarrow$ vb (since $\dim \bar{S} = 1$ and $\bar{S}$ smooth). ∎

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3) Local monodromy theorem

$X_{/k}$ smooth connected curve, $D \subset X$ divisor

$\mathcal{E}_{/X}$ v.bun., $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1(\log D)$

$p \in D$, x coordinate at p

__Def__ $\mathrm{Res}_p \nabla \in \mathrm{End}_k(\mathcal{E}(p))$ induced by

$\nabla_{x\frac{d}{dx}} : \mathcal{E}_p \to \mathcal{E}_p$.

__Note:__ does not dep. on x! $x' = ux$, $u \in \mathcal{O}^\times_{X_p}$

$\Rightarrow x'\frac{d}{dx'} = (1 + u^{-1}xf)^{-1} x\frac{d}{dx}$, $du = f dx$

<u>Ex</u> Locally $\nabla = d + \Omega$, $\Omega = (\omega_{ij}) = (f_{ij} \frac{dx}{x})$

$\rightsquigarrow \mathrm{Res}_p \nabla = (f_{ij}(0))_{ij}$

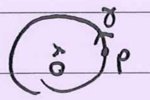
<u>Ex</u> Over D^*, $z \frac{d}{dz} + \alpha = 0 \rightsquigarrow \nabla = d + \alpha \frac{dz}{z}$

$\rightsquigarrow \mathrm{Res}_0 \nabla = \alpha$. Recall: monodromy $e^{-2\pi i \alpha}$

<u>Prop.</u> (Residues and monodromy)

holomorphic

Let $\nabla : \mathcal{O}_D^{\oplus r} \to \mathcal{O}_D^{\oplus r} \otimes \Omega^1_D(\log 0)$, $p \in D^*$,



$T \in GL(\mathcal{O}_D^{\oplus r}(p)) \cong GL_r(\mathbb{C})$ monodromy ass. to $\nabla|_{D^*}$.

Then $T \overset{=}{\cancel{\text{and}}} \exp(2\pi i \, \mathrm{Res}\, \nabla)$ ~~are conjugate~~.

Proof: Suppose $\nabla = d + \Omega$, where $\Omega = A \frac{dz}{z}$

$A \overset{= \mathrm{Res}\,\nabla}{\in} M_{n \times n}(\mathbb{C})$ cst. $\Rightarrow$ colums of $\exp(-A \log z)$

form a fundamental system of soln's. Monodromy:

$\exp(-A(\log z + 2\pi i)) = \exp(-A \log z) \exp(-2\pi i A)$.

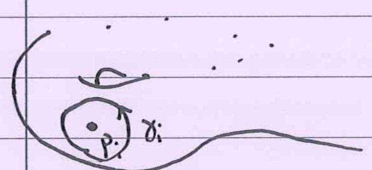
General case is similar. ■

<u>Thm</u> (Local monodromy thm) $\pi : X \to S$ proper smooth, $X, S/\mathbb{C}$ smooth, connected,

$\dim S = 1$. ~~Let $\bar{S} \supset S$ be the smooth compact. of S.~~

Then $\forall n$, $R^n \pi_*^{an} \mathbb{C}_{X^{an}}$ has quasi-unipotent monodromy at every pt of ∞.

$\bar S \supset S$ smooth compactification, $\bar S = S \cup \{p_1, \dots, p_n\}$



T_i "monodromy of p_i"

$\Rightarrow \exists\, N$ st $T_i^N - I$ nilpotent

($\Leftrightarrow$ $\forall i$ eigenvalues of T_i are roots of unity)

Exercise For $\lambda(1-\lambda)\dfrac{d^2}{d\lambda^2} + (1-2\lambda)\dfrac{d}{d\lambda} - \dfrac{1}{4} = 0$

eigenvalues of monodromy at ∞ : $\{i, i\}$

Proof (Brieskorn) : • $R^n \pi_*^{an} \mathbb{C}_{X^{an}} = R^n \pi_*^{n} \mathbb{Q}_{X^{an}} \otimes \mathbb{C}_{X^{an}}$

$\Rightarrow$ can assume $T_i \in M_{r\times r}(\mathbb{Q})$.

• $R^n \pi_*^{an} \mathbb{C}_{X^{an}}$ = horizontal sections of $(\mathcal{H}^n_{dR}(X/S), \nabla)$

Regularity $\Rightarrow$ $(\mathcal{H}^n, \nabla)$ extends to $(\bar{\mathcal{H}}^n, \bar\nabla)$ over $\bar S$

$\Rightarrow$ T_i conjugate to $\exp(-2\pi i \underbrace{R_i}_{\mathrm{Res}_{p_i}\bar\nabla})$

• $\forall\, \sigma \in \mathrm{Aut}(\mathbb{C})$

$$\begin{tikzcd}
X^\sigma \arrow[r] \arrow[d, "\pi^\sigma"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "\pi"] \\
S^\sigma \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & S \arrow[d] \\
\mathrm{Spec}\,\mathbb{C} \arrow[r, "\mathrm{Spec}\,\sigma"'] & \mathrm{Spec}\,\mathbb{C}
\end{tikzcd}$$

~~monodromy at p_i^σ : T_i^σ~~

GM compatible w/ base change

$\Rightarrow$ $\mathrm{Res}_{p_i^\sigma} \underbrace{\bar\nabla^\sigma}_{\text{GM of } \pi^\sigma} = \exp(-2\pi i R_i^\sigma)$

- Thus $\forall \sigma \in \mathrm{Aut}(\mathbb{C})$, $\exp(-2\pi i R_i^{\sigma})$ is conjugate to a matrix in $M_{r\times r}(\mathbb{Q})$

λ eigenvalue of $R_i \Rightarrow e^{-2\pi i \lambda^{\sigma}} \in \bar{\mathbb{Q}}$ $\forall \sigma \in \mathrm{Aut}(\mathbb{C})$

In part., $\lambda \in \bar{\mathbb{Q}}$ (otherwise, $\exists\ \sigma : \lambda \mapsto -\frac{1}{2\pi i}$)

Thus $\lambda, e^{2\pi i \lambda} \in \bar{\mathbb{Q}} \Longrightarrow \lambda \in \mathbb{Q}$. ∎

↳ Gelfond - Schneider thm:

$a, b \in \bar{\mathbb{Q}} \Rightarrow a^b \notin \bar{\mathbb{Q}}$ unless $a = 0, 1$ or $b \in \mathbb{Q}$

$u \in \mathbb{C} \setminus 2\pi i \mathbb{Z}$, $b \in \bar{\mathbb{Q}}$ st

$$e^u, e^{bu} \in \bar{\mathbb{Q}} \Rightarrow b \in \mathbb{Q}$$

Take $u = \pi i$, $b = 2\lambda$