Calculus on Schemes - Exercises

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Exercise 0.1. Let A be an R-algebra, and M be an A-module. Let $D \in \mathrm{Der}_R(A, M)$, $n \in \mathbf{N}$, and $f_1, \ldots, f_n \in A$. Prove that for every $n \in \mathbf{N}$

$$D(f_1 \cdots f_n) = \sum_{i=1}^{n} \left(\prod_{j \neq i} f_j \right) D(f_j).$$

In particular, for every $f \in A$,

$$D(f^n) = n f^{n-1} D(f).$$

If f is invertible in A, then the above formula also works for negative n.

Exercise 0.2. Let A be an R-algebra.

1. Prove that the composition of derivations in $\mathrm{Der}_R(A)$ is not in general a derivation (i.e., give some simple counterexamples).

2. For $D \in \mathrm{Der}_R(A)$, we set

$$D^n = \underbrace{D \circ D \circ \cdots \circ D}_{n \text{ times}} \in \mathrm{End}_R(A)$$

   Prove that for every $f, g \in A$, we have

$$D^n(fg) = \sum_{j=0}^{n} \binom{n}{j} D^j(f) D^{n-j}(g).$$

   Conclude that, if R is of characteristic p, then $D^p \in \mathrm{Der}_R(A)$ for every $D \in \mathrm{Der}_R(A)$. Can you compute $(D_1 + D_2)^p$ in terms of D_1 and D_2? What about $(fD)^p$?

3. Let $D_1, D_2 \in \mathrm{Der}_R(A)$. The Lie bracket $[D_1, D_2] \in \mathrm{End}_R(A)$ is defined by the usual commutator:

$$[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1.$$

   Prove that $[D_1, D_2] \in \mathrm{Der}_R(A)$.

4. Polynomial expressions in arbritraty compositions of derivations are known as "differential operators". Let $D = \frac{d}{dx} \in \mathrm{Der}_R(R[x])$. Prove, for every $n \in \mathbf{N}$, the following identity of differential operators:

$$x^{n+1} D^{n+1} = xD \circ (xD - 1) \circ (xD - n)\,.$$

Exercise 0.3. Let L be a finite field extension of K. Prove that L/K is separable if and only if $\Omega^1_{L/K} = 0$.

Exercise 0.4. Let $p : \mathbb{V}(\mathcal{E}) \longrightarrow S$ be the total space of a vector bundle $\mathcal{E}$ over a scheme S. Prove that

$$\Omega^1_{E/S} \cong p^*\mathcal{E}$$

canonically. Hint: prove first that for any R-modules M and N, we can identify $\mathrm{Hom}_R(M,N) = \mathrm{Der}_R(\mathrm{Sym}\, M, N)$ and conclude that $\mathrm{Sym}\, M \otimes_R M \longrightarrow \Omega^1_{\mathrm{Sym}\, M/R}$, $f \otimes m \longmapsto f dm$, is an isomorphism of $\mathrm{Sym}\, M$-modules.

Exercise 0.5. Let k be a field, $f \in k[x,y]$, and $X = \mathrm{Spec}\, k[x,y]/(f)$ be the affine plane curve defined by f. Assume that f, $\partial f/\partial x$, and $\partial f/\partial y$ are coprime, and consider the open subset $U = D(\partial f/\partial y)$ of X. Prove that the differential form $(\partial f/\partial y)^{-1} dx \in \Gamma(U, \Omega^1_{X/k})$ extends to a global section ω of $\Omega^1_{X/k}$ and that $\Omega^1_{X/k} = \mathcal{O}_X \omega$.

Exercise 0.6.

1. Consider a Cartesian square of schemes

$$\begin{array}{ccc} X' & \xrightarrow{\varphi} & X \\ \downarrow & \square & \downarrow \\ S' & \longrightarrow & S \end{array}$$

   Prove that we have a canonical isomorphism

$$\Omega^1_{X'/S'} \cong \varphi^* \Omega^1_{X/S}.$$

2. Let X_1 and X_2 be S-schemes, prove that

$$\Omega^1_{X_1 \times_S X_2/S} = \Omega^1_{X_1 \times_S X_2/X_2} \oplus \Omega^1_{X_1 \times_S X_2/X_1}.$$

   In particular, if $p_i : X_1 \times_S X_2 \longrightarrow X_i$ denotes the ith projection, $i = 1, 2$, then conclude from 1 that we have a canonical isomorphism

$$\Omega^1_{X_1 \times_S X_2/S} \cong p_1^* \Omega^1_{X_1/S} \oplus p_2^* \Omega^1_{X_2/S}.$$

Exercise 0.7. Let R be a ring. Describre all the vector fields on $\mathbf{A}^1_R$ that lift to a vector field on $\mathbf{P}^1_R$.

Exercise 0.8. Let R be a ring. Prove that there is a canonical exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^n_R} \longrightarrow \mathcal{O}_{\mathbf{P}^n_R}(1) \longrightarrow T_{\mathbf{P}^n_R/R} \longrightarrow 0.$$

Hint: see Hartshorne II.8.13 (can you find a different proof?).

Exercise 0.9. Let $\pi : X \longrightarrow S$ be a morphism of schemes, and define the *Picard functor* $\mathrm{Pic}_{X/S}$ by

$$\mathrm{Pic}_{X/S}(T) = \mathrm{Pic}(X \times_S T)/\mathrm{Pic}(T) \qquad (T \in \mathsf{Sch}_{/S})$$

We say that $\pi_*\mathcal{O}_X = \mathcal{O}_S$ holds universally if $(\pi_T)_*\mathcal{O}_{X \times_S T} = \mathcal{O}_T$ for every S-scheme T. For instance, this holds if π is proper, flat, surjective and with geometrically integral fibers. Now, let $S = \mathrm{Spec}\, k$ where k is a field, assume that $\pi_*\mathcal{O}_X = \mathcal{O}_S$ holds universally, and that $\mathrm{Pic}_{X/k}$ is representable by a smooth k-scheme. Let $e \in \mathrm{Pic}_{X/k}(k) = \mathrm{Pic}(X)$ be given by the trivial line bundle $\mathcal{O}_X$ on X. Prove that there's a natural isomorphism of k-vector spaces

$$T_{\mathrm{Pic}_{X/k}/k}(e) = H^1(X, \mathcal{O}_X).$$

In particular, the tangent space at the origin of an elliptic curve E is naturally isomorphic to $H^1(E, \mathcal{O}_E)$. In general, if A is an abelian variety over k, and $A^\vee$ denotes the dual abelian variety, then the tangent space at the origin of $A^\vee$ is naturally isomorphic to $H^1(A, \mathcal{O}_A)$.

Exercise 0.10. Let R be a ring and $f \in R[x_1, \ldots, x_n]$. Prove that the the R-scheme $X = \operatorname{Spec} R[x_1, \ldots, x_n]/(f)$ is smooth if and only if

$$\left(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right) = (1)$$

as ideals in $R[x_1, \ldots, x_n]$.

Exercise 0.11. Let $\pi_n : \mathbf{A}^1_{\mathbf{C}} \longrightarrow \mathbf{A}^1_{\mathbf{C}}$ be defined by $\pi_n^*(t) = t^n$. Prove that π_n is unramified if and only if $n = 1$.

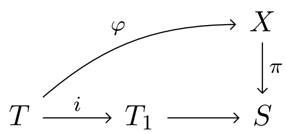
Exercise 0.12. Let k be a field and $X = \operatorname{Spec} k[x, y, z]/(xz - y)$.

1. Prove that X is a smooth algebraic variety over k.
2. Let $\pi : X \longrightarrow \mathbf{A}^2_k$ be the morphism of k-schemes induced by the inclusion $k[x, y] \longrightarrow k[x, y, z]/(xz - y)$. Prove that every geometric fiber of π is smooth.
3. Prove that π is *not* smooth.
4. Here, X is an affine open subset of the blow-up of $\mathbf{A}^2_k$ at the origin. Generalize the above facts to more general blow-ups.

Exercise 0.13. Let R be a ring. Recall that a standard étale algebra over R is an R-algebra of the form $A = R[x]_g/(f)$, where $f, g \in R[x]$, f is monic and the image of df/dx in A is a unit. Prove directly from the definition that if A is a standard étale algebra over R, then $\operatorname{Spec} A \longrightarrow \operatorname{Spec} R$ is étale.

Exercise 0.14. Let $\pi : X \longrightarrow S$ be étale at $p \in X$. Prove that $\hat{\mathcal{O}}_{X,p}$ is a finite $\hat{\mathcal{O}}_{S,\pi(p)}$ algebra isomorphic to a finite direct sum $\bigoplus_{i=1}^n \hat{\mathcal{O}}_{S,\pi(p)}$ as an $\hat{\mathcal{O}}_{S,\pi(p)}$-module. Hint: this is a bit like Hensel's lemma.

Exercise 0.15. Let $\pi : X \longrightarrow S$ be a morphism of schemes, and consider a diagram



where $i : T \longrightarrow T_1$ is a thickening.

1. Prove that
$$U \longmapsto \Gamma(U, \mathcal{F}) := \{\psi \in \operatorname{Hom}_S(i(U), X) \mid \psi \circ i|_U = \varphi|_U\}$$
is a sheaf of sets on T.
2. Show that the sheaf of $\mathcal{O}_T$-modules
$$\mathcal{H} := \mathcal{H}om_{\mathcal{O}_T}(\varphi^* \Omega^1_{X/S}, C_{T/T_1})$$
acts on $\mathcal{F}$, and that whenever $\Gamma(U, \mathcal{F}) \neq \emptyset$, the action
$$\Gamma(U, \mathcal{H}) \times \Gamma(U, \mathcal{F}) \longrightarrow \Gamma(U, \mathcal{F})$$
is simply transitive. Hint: consider first the case where everything is affine to understand what's happening.

3. Show that if π is locally smooth, then $\mathcal{F}$ is a $\mathcal{H}$-torsor.

4. Conclude that every locally smooth morphism is smooth. Hint: use that H^1 of a quasi-coherent sheaf on an affine scheme vanishes.

Exercise 0.16. Prove that a morphism of schemes $\pi : X \longrightarrow S$ is unramified at $p \in X$ if and only if $\mathfrak{m}_{\pi(p)}\mathcal{O}_{X,p} = \mathfrak{m}_p$ and $k(p)$ is a separable extension of $k(\pi(p))$.

Exercise 0.17. Prove that a morphism of schemes if étale if and only if it is flat and unramified.

Exercise 0.18. Let $\varphi : X \longrightarrow Y$ be a morphism of smooth algebraic varieties over a field k. Prove that if φ is finite, unramified, and universally injective, then it is a closed immersion.

Exercise 0.19. Let k be a field, and X be a smooth, projective and geometrically connected curve over k. Assume that X is of genus $g = 1$, and that X has a rational point $p \in X(k)$. Let us prove that X is isomorphic to a projective plane curve given by a 'generalized Weierstrass equation':

$$y^2z + a_1xyz + a_3yz^2 = x^3 + a_2x^2z + a_4xz^2 + a_6z^3$$

1. Prove that $\Omega^1_{X/k}$ is trivial.

2. Prove that $\dim_k H^0(X, \mathcal{O}_X(D)) = \deg D$

3. Fix a local coordinate t at p and let ω be the unique global section of $\Omega^1_{X/k}$ such that $\omega = (1 + O(t))dt$ in a 'formal neighborhood' of p. Prove that there exists $x \in H^0(X, \mathcal{O}_X(2[p])$ and $y \in H^0(X, \mathcal{O}_X(3[p]))$ such that $(1, x, y)$ is a basis of $H^0(X, \mathcal{O}_X(3[p]))$. Conclude that x and y (seen as rational functions on X) necessarily satisfy an equation of the form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

for some $a_i \in k$.

4. Show that $\mathcal{O}_X(3[p])$ is very ample and conclude. Hint: apply Exercise 0.18 (cf. Liu Prop. 7.4.5).

Exercise 0.20. Let k be a field and consider a hypterelliptic curve X over k defined by $y^2 = f(x)$, where $f \in k[x]$ is of degree d. Recall that the coordinates at infinity are given by $(t, s) = (1/x, y/x^e)$, where $d = 2e$ or $d = 2e - 1$ (depending on the parity of d). Prove that

$$\frac{dx}{y}, \; x\frac{dx}{y}, \; \ldots x^{e-2}\frac{dx}{y}$$

gives a basis of $H^0(X, \Omega^1_{X/k})$, so that X is of genus $e - 1$.

Exercise 0.21. Let k be an algebraically closed field, and X be a smooth, projective and connected curve over k. Let $p \in X$ be a closed point and fix a local coordinate x at p, so that $\hat{\mathcal{O}}_{X,p} = k[\![x]\!]$. For a meromorphic differential ω on X, we can write

$$\omega = \left(\sum_{n \gg -\infty} a_n x^n \right) dx$$

in a 'formal neighborood' of p. We define its *residue* at p by

$$\mathrm{res}_p(\omega) = a_{-1} \in k.$$

A classical theorem asserts that for any meromorphic differential, we have $\sum_{p \in X} \mathrm{res}_p(\omega) = 0$.

1. Prove that $\mathrm{res}_p(\omega)$ does not depend on the choice of x.

2. Let $D = \sum_{i=1}^{r} n_i[p_i]$ be a divisor on X. Prove that connections ∇ on $\mathcal{O}_X(D)$ correspond to meromorphic differentials $\omega \in H^0(X, \mathcal{O}_X([p_1] + \cdots [p_r]))$ such that $\mathrm{res}_{p_i}(\omega) = n_i$.

3. Prove that a line bundle $\mathcal{L}$ on X admits a connection if and only if $\deg \mathcal{L} = 0$.

Exercise 0.22. Prove that the complex unit disk is not algebraic, i.e., there's no **C**-scheme locally of finite type X such that X^{an} is isomorphic to $\{z \in \mathbf{C} \mid |z| < 1\}$.