A crash course in modular forms and cohomology - Lecture 1

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1 Modular forms

We denote the upper half-plane by $\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$. The group $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{H}$ by fractional linear transformations:

$$\gamma\tau = \frac{a\tau + b}{c\tau + d}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Note that the formula

$$\mathrm{Im}(\gamma\tau) = \frac{\mathrm{Im}(\tau)}{|c\tau + d|^2}$$

guarantees that we indeed have $\gamma\tau \in \mathbb{H}$.

Definition 1.1. Let k be an integer. A *modular form* of weight k is a holomorphic function $f : \mathbb{H} \longrightarrow \mathbb{C}$ satisfying

$$f(\gamma\tau) = (c\tau + d)^k f(\tau), \qquad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

and which bounded as $\mathrm{Im}(\tau) \to +\infty$.

There are no non-zero modular forms of odd weight (consider $-I_2 \in \mathrm{SL}_2(\mathbb{Z})$). It is a theorem that there are no non-zero modular forms of weight $k \leq 2$ ([3] Chapter 7, Theorem 4). For every other k, we always have non-trivial examples of modular forms.

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Example 1 (Eisenstein series)**.** For all even $k \geq 4$, the *Eisenstein series*

$$G_k(\tau) := \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(m+n\tau)^k}$$

is a modular form of weight k. Moreover,

$$\lim_{\text{Im}\tau\to+\infty} G_k(\tau) = \sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{1}{m^k} = 2\zeta(k)$$

where ζ is Riemann's zeta function.

By taking combinations of Eisenstein series, we can form other modular forms.

Example 2 (Ramanujan's delta function)**.** The function

$$\Delta(\tau) = \frac{1}{1728}\left(\left(\frac{G_4(\tau)}{2\zeta(4)}\right)^3 - \left(\frac{G_6(\tau)}{2\zeta(6)}\right)^2\right)$$

is a modular form of weight 12.

As every modular form f is invariant under $\tau \longmapsto \tau+1$ and is bounded as $\text{Im}(\tau) \to +\infty$, it can be written as

$$f(\tau) = a_0 + a_1 q + a_2 q^2 + a_3 q^3 + \cdots, \qquad q = e^{2\pi i\tau},$$

for unique $a_n \in \mathbb{C}$. For example, one can show that Fourier expansion of an Eisenstein series is given by

$$G_k(\tau) = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{n\geq 1}\sigma_{k-1}(n)q^n$$

where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ ([3] Chapter 7, Proposition 8).

Definition 1.2. A *cuspform* is a modular form f for which $a_0 = 0$.

For instance, Ramanujan's delta is a cuspform:

$$\Delta(\tau) = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - \cdots \in \mathbb{Z}[\![q]\!].$$

As a motivation, let us further explore this example. The Fourier coefficients a_n of Δ can be shown to satisfy the following properties:

- $a_{mn} = a_m a_n$ for every integers $m, n \geq 1$ satisfying $\gcd(m,n) = 1$;
- $a_{p^{r+1}} = a_p a_{p^r} - p^{11} a_{p^{r+1}}$ for every prime p and every integer $r \geq 0$.

Equivalently, these identities mean that the Dirichlet series

$$L(\Delta, s) := \sum_{n=1} \frac{a_n}{n^s}$$

admits the following Euler product

$$L(\Delta, s) = \prod_p \frac{1}{1 - a_p p^{-s} + p^{11-2s}}.$$

The series $L(\Delta, s)$ admits an analytic continuation to an entire function of $s \in \mathbb{C}$ and satisfies the following aditional properties:

1. For every integer $0 \leq m \leq 12$, we have

$$L(\Delta, m) = \frac{(2\pi)^m}{(m-1)!} \int_0^\infty \Delta(it) t^{m-1} dt.$$

   The above integrals are examples of *periods of modular forms*, and can be shown to be expressible as an integral of a rational function with rational coefficients, i.e., *a Kontsevich-Zagier period.*

2. The above Euler product "comes from a family of ℓ-adic representations"

$$\rho_{\Delta,\ell} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GL}_2(\mathbb{Q}_\ell).$$

   This means that for each prime $p \neq \ell$, the characteristic polynomial of $\rho(\mathrm{Frob}_p)$ is $1 - a_p X + p^{11} X^2$.

Both the above properties are related to algebraic geometry, and indeed have a common cause: the existence of a 'motive' M_Δ attached to Δ, whose 'realisations' are suitable pieces of certain cohomology groups $H^1(\mathcal{M}_{1,1}, \mathrm{Sym}^{10}\mathcal{H})$. The purpose of these lectures is to give an introduction to these objects.

2 Complex tori

Definition 2.1. A subgroup Λ of $(\mathbb{C}, +)$ is a *lattice* if it is discrete and if the quotient $\mathbb{C}/\Lambda$ is compact.

It is an exercise to prove that any lattice in $\mathbb{C}$ is of the form

$$\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2,$$

for some $\omega_1, \omega_2 \in \mathbb{C}$ linearly independent over $\mathbb{R}$.

Given a lattice $\Lambda \subset \mathbb{C}$, the quotient $\mathbb{C}/\Lambda$ is a *compact Riemann surface*, the complex charts being given by local sections of the quotient map $\mathbb{C} \longrightarrow \mathbb{C}/\Lambda$, with a *complex Lie group structure* induced by the addition on $\mathbb{C}$:

$$(z + \Lambda) + (w + \Lambda) = (z + w) + \Lambda.$$

These are the properties characterising complex tori.

Definition 2.2. A *complex torus* is a compact Riemann surface with the structure of a complex Lie group. A morphism of complex tori is a morphism of complex Lie groups.

We have seen that any lattice $\Lambda \subset \mathbb{C}$ gives rise to a complex torus $\mathbb{C}/\Lambda$. In fact, every complex torus is of the form.

Theorem 2.3. *If X is a complex torus, then there exists a lattice $\Lambda \subset \mathbb{C}$ such that X is isomorphic to $\mathbb{C}/\Lambda$.*

We only explain the main ideas of the proof; details can be found in [1, I.1]. As for any complex Lie group, we can consider the vector space $\mathrm{Lie}\, X$, defined as the tangent space $T_e X$ at the identity $e \in X$, and the exponential map

$$\exp : \mathrm{Lie}\, X \longrightarrow X,$$

a local biholomorphism at 0, sending 0 to e. Compactness of X allows to prove that the group law is commutative, so that exp is also a morphism of complex Lie groups. Further, exp must be surjective as its image contains a neighborhood of $e = \exp(0)$, and its kernel discrete as it's injective in a neighborhood of 0. This proves the theorem, since $\operatorname{Lie} X \cong \mathbb{C}$ and $\ker(\exp)$ is a lattice in $\operatorname{Lie} X$.

We can be more precise. By using the identification

$$\operatorname{Lie} X \cong H^0(X, \Omega^1)^\vee$$

induced by the natural duality between tangent and cotangent vectors, we can prove that $\ker(\exp)$ is the image of the map $H_1(X, \mathbb{Z}) \longrightarrow \operatorname{Lie} X$ sending γ to $(\omega \longmapsto \int_\gamma \omega)$. Thereby, we obtain an exact sequence

$$0 \longrightarrow H_1(X, \mathbb{Z}) \longrightarrow \operatorname{Lie} X \xrightarrow{\exp} X \longrightarrow 0, \tag{1}$$

which is just a fancy version of

$$0 \longrightarrow \Lambda \longrightarrow \mathbb{C} \longrightarrow \mathbb{C}/\Lambda \longrightarrow 0.$$

Remark 1. Concretely, if $X = \mathbb{C}/\Lambda$, then $H^0(X, \Omega^1) = \mathbb{C}[dz]$ and the identification of the lattice Λ with $H_1(X, \mathbb{Z})$ is obtained by associating, to every $\lambda \in \Lambda$, the homology class of the loop $[0, 1] \longrightarrow X$ sending t to $t\lambda + \Lambda$.

The exact sequence (1) is functorial, meaning that any morphism of complex tori $\varphi : X' \longrightarrow X$ yields a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_1(X', \mathbb{Z}) & \longrightarrow & \operatorname{Lie} X' & \longrightarrow & X' & \longrightarrow & 0 \\ & & \big\downarrow \varphi_* & & \big\downarrow \operatorname{Lie}\varphi & & \big\downarrow \varphi & & \\ 0 & \longrightarrow & H_1(X, \mathbb{Z}) & \longrightarrow & \operatorname{Lie} X & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

where $\operatorname{Lie} \varphi = d\varphi_e$, the differential of φ at $e \in X'$, and φ_* is the pushforward in homology. As $\operatorname{Lie} \varphi$ is linear, we immediately obtain the following characterisation.

Lemma 2.4. *If Λ and Λ' are lattices in $\mathbb{C}$, then*

$$\{\alpha \in \mathbb{C} \mid \alpha\Lambda' \subset \Lambda\} \longrightarrow \operatorname{Hom}(\mathbb{C}/\Lambda', \mathbb{C}/\Lambda), \qquad \alpha \longmapsto [\alpha] : z + \Lambda' \longmapsto \alpha z + \Lambda$$

is a bijection. □

It follows that two lattices Λ and Λ' give rise to isomorphic complex tori if and only if they are homothetic: there exists $\alpha \in \mathbb{C}$ such that $\alpha\Lambda' = \Lambda$.

Example 3. For any $\tau \in \mathbb{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im}(\tau) > 0\}$, we set

$$\mathbb{X}_\tau = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau).$$

Every complex torus is isomorphic to some $\mathbb{X}_\tau$. Indeed, given a lattice $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, up to swapping ω_1 and ω_2, we can assume that $\tau = \omega_2/\omega_1$ has positive imaginary part. It follows from

$$\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \omega_1(\mathbb{Z} + \mathbb{Z}\tau)$$

that $\mathbb{C}/\Lambda$ is isomorphic to $\mathbb{X}_\tau$.

Example 4. Given $\tau, \tau' \in \mathbb{H}$, when are $\mathbb{X}_\tau$ and $\mathbb{X}_{\tau'}$ isomorphic? A morphism $\varphi : \mathbb{X}_{\tau'} \longrightarrow \mathbb{X}_\tau$ corresponds to $\alpha \in \mathbb{C}$ satisfying

$$\alpha \begin{pmatrix} \tau' \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{\gamma} \begin{pmatrix} \tau \\ 1 \end{pmatrix}$$

for some $a, b, c, d \in \mathbb{Z}$. By dividing the first row by the second row, we get

$$\tau' = \frac{a\tau + b}{c\tau + d}.$$

Since

$$\operatorname{Im}(\tau') = (ad - bc)\frac{\operatorname{Im}(\tau)}{|c\tau + d|^2},$$

we obtain $\det(\gamma) = ad - bc > 0$. If φ is an isomorphism, then $\gamma \in \mathrm{GL}_2(\mathbb{Z})$, but the positivity of the determinant actually implies that $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

Conversely, if $\tau' = \gamma\tau$ with $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, then the multiplication by $c\tau + d$ gives an isomorphism $\mathbb{X}_{\tau'} \cong \mathbb{X}_\tau$. We conclude that $\mathbb{X}_\tau$ is isomorphic to $\mathbb{X}_{\tau'}$ if and only if τ and τ' are equivalent under the left action $(\gamma, \tau) \longmapsto \gamma\tau$ of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$.

3 The moduli stack of complex tori

Is there a space classifying complex tori? The answer depends on what we understand by 'space' and by 'classifying'. It follows from the above examples that the left quotient $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ is in bijection with isomorphy classes of complex tori.

Despite the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ not being free, the quotient can be given a natural complex structure and can be shown to be isomorphic to $\mathbb{C}$. We say that $\mathbb{C}$ is a *coarse moduli space* of complex tori. This is good enough for some applications, but not for modular forms.

In order to get finer notions of moduli, we need to work with families.

Definition 3.1. Let S be a complex manifold. A *family of complex tori* over S is a proper holomorphic map $p : X \longrightarrow S$ with the structure of a relative complex Lie group over S such that each fibre of p is a complex torus.

The definition of a relative complex Lie group is analogous to that of a group scheme: there is a holomorphic section $e : S \longrightarrow X$ of p (identity), and holomorphic maps $\mu : X \times_S X \longrightarrow X$ and $\iota : X \longrightarrow X$ over S (multiplication and inverse) satisfying the usual group axioms.

Example 5. Consider the following action of $\mathbb{Z}^2$ on $\mathbb{C} \times \mathbb{H}$:

$$(m, n) \cdot (z, \tau) = (z + m + n\tau, \tau).$$

One can show that this action is proper and free, so that the quotient is a 2-dimensional complex manifold $\mathbb{X}$. The projection $\mathbb{C} \times \mathbb{H} \longrightarrow \mathbb{H}$ induces a holomorphic map $p : \mathbb{X} \longrightarrow \mathbb{H}$, which admits the section $e : \mathbb{H} \longrightarrow \mathbb{X}$ sending τ to $[(0, \tau)]$. One can check that the holomorphic maps

$$\mu : \mathbb{X} \times_{\mathbb{H}} \mathbb{X} \longrightarrow \mathbb{X}, \qquad ([(z_1, \tau)], [(z_2, \tau)]) \longmapsto [(z_1 + z_2, \tau)]$$

and

$$\iota : \mathbb{X} \longrightarrow \mathbb{X}, \qquad [(z, \tau)] \longmapsto [(-z, \tau)]$$

are well-defined and give $p : \mathbb{X} \longrightarrow \mathbb{H}$ the structure of a relative complex Lie group over $\mathbb{H}$. In fact, p is a family of complex tori over $\mathbb{H}$: for any $\tau \in \mathbb{H}$ we have $p^{-1}(\tau) = \mathbb{X}_\tau$ (see Example 3).

Definition 3.2 (Moduli stack of complex tori)**.** We define a category $\mathcal{M}^{\mathrm{an}}$ as follows. Its objects are given by families of complex tori $p : X \longrightarrow S$ over arbitrary complex manifolds S, and its morphisms are given by Cartesian squares (i.e., pullback diagrams)

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \big\downarrow & \square & \big\downarrow \\ S' & \longrightarrow & S \end{array}$$

The category $\mathcal{M}^{\mathrm{an}}$ comes with a natural forgetful functor to the category of complex manifolds

$$\mathcal{M}^{\mathrm{an}} \longrightarrow (\mathbb{C}\text{-manifolds}), \qquad (p : X \longrightarrow S) \longmapsto S$$

giving it the structure of a *category over the category of complex manifolds.*

The *fibre* over a complex manifold S is the full subcategory $\mathcal{M}^{\mathrm{an}}(S)$ of objects of $\mathcal{M}^{\mathrm{an}}$ mapping to S.

Remark 2. In fact, $\mathcal{M}^{\mathrm{an}}$ is *category fibred in groupoids* over the category of complex manifolds, meaning that we can make sense of pullbacks and that each fibre category is a groupoid (a category in which every morphism is an isomorphism). Further, we can put a Grothendieck topology on the category of complex manifolds in such a way that $\mathcal{M}^{\mathrm{an}}$ satisfies a 'sheaf condition'; this means that $\mathcal{M}^{\mathrm{an}}$ is a *stack.* This justifies the name 'moduli stack'. We refer to [2, Chapters III and IV] for the general definitions.

A 'fine moduli space' S of complex tori, if it existed, should be the base of a terminal object of $\mathcal{M}^{\mathrm{an}}$, i.e., a family of complex tori $X \longrightarrow S$ that is *universal*, in the sense that every other family $X' \longrightarrow S'$ is a pullback of $X \longrightarrow S$ via a unique map $S' \longrightarrow S$. Such terminal object cannot exist, because unicity will always be violated. Indeed, every family of complex tori $X \longrightarrow S$ admits a non-trivial automorphism in $\mathcal{M}^{\mathrm{an}}$, namely, multiplication by -1:

$$\begin{array}{ccc} X & \xrightarrow{[-1]} & X \\ \big\downarrow & \square & \big\downarrow \\ S & \xrightarrow[\mathrm{id}]{} & S \end{array}$$

We can remedy this in two ways: by introducing 'level structures' (see next lecture), or by treating $\mathcal{M}^{\mathrm{an}}$ as a space in its own right.

To explain the second approach, we introduce two constructions.

Example 6 (Complex manifolds as stacks)**.** To every complex manifold S we can associate a category $\underline{S}$ over ($\mathbb{C}$-manifolds) as follows. The objects of $\underline{S}$ are complex manifolds over S, that is, holomorphic maps $T \longrightarrow S$. Morphisms are given by commutative diagrams

$$\begin{array}{ccc} T' & \longrightarrow & T \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

The '2-Yoneda lemma' guarantees that S is determined by $\underline{S}$. Thus, by abuse, we can drop the underline in the notation and always see a complex manifold S as a category over ($\mathbb{C}$-manifolds) defined as above.

By a *morphism* between two categories over ($\mathbb{C}$-manifolds) we mean a functor commuting with the projections to ($\mathbb{C}$-manifolds). The next lemma formalises the idea that $\mathcal{M}^{\mathrm{an}}$ plays the role of a fine moduli space of complex tori.

Lemma 3.3. *Let S be a complex manifold. Then, morphisms $S \longrightarrow \mathcal{M}^{\mathrm{an}}$ correspond to families of complex tori $X \longrightarrow S$.*

Proof. To a family of complex tori $X \longrightarrow S$, we associate the functor which sends a complex manifold T over S to the pullback family of complex tori $X \times_S T \longrightarrow T$. Conversely, to a functor $S \longrightarrow \mathcal{M}^{\mathrm{an}}$ over ($\mathbb{C}$-manifolds), we associate the family $X \longrightarrow S$ given by the image of $\mathrm{id} : S \longrightarrow S$. These constructions are inverse of each other. □

Example 7 (Stacky quotient or orbifold quotient)**.** Let G be a discrete group with a proper left action on a complex manifold Y. We define a category $[G\backslash Y]$ over ($\mathbb{C}$-manifolds) as follows. Objects are principal G-bundles $P \longrightarrow S$ with a G-equivariant morphism $P \longrightarrow S$:

$$\begin{array}{ccc} P & \longrightarrow & Y \\ \downarrow & & \\ S & & \end{array}$$

Morphisms are morphisms of G-bundles compatible with the maps to Y. Note that there is a canonical morphism $Y \longrightarrow [G\backslash Y]$ sending $f : S \longrightarrow Y$ to the trivial bundle $G \times S \longrightarrow S$ with the equivariant map $G \times S \longrightarrow Y$ given by $(g, t) \longmapsto g \cdot f(t)$.

This allow us to uniformise the moduli stack of complex tori.

Theorem 3.4. *The morphism $\pi : \mathbb{H} \longrightarrow \mathcal{M}^{\mathrm{an}}$ corresponding to the family $\mathbb{X} \longrightarrow \mathbb{H}$ of Example 5 factors through $\mathbb{H} \longrightarrow [\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}]$ and defines an equivalence of categories*

$$[\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}] \rightleftarrows \mathcal{M}^{\mathrm{an}}.$$

Sketch of proof. Let $(P \longrightarrow S, P \longrightarrow \mathbb{H})$ be an object of $[\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}]$. Since $P \longrightarrow S$ is locally trivial, we can find an open covering $S = \bigcup_i S_i$ with sections $\sigma_i : S_i \longrightarrow P$. By composing with the $\mathrm{SL}_2(\mathbb{Z})$-equivariant map $P \longrightarrow \mathbb{H}$, we get maps $f_i : S_i \longrightarrow \mathbb{H}$. Let $X_i \longrightarrow S_i$ be the family of complex tori defined as the pullback

$$\begin{array}{ccc} X_i & \xrightarrow{\varphi_i} & \mathbb{X} \\ \downarrow & \square & \downarrow \\ S_i & \xrightarrow[f_i]{} & \mathbb{H} \end{array}$$

We now observe that the $X_i \longrightarrow S_i$ glue into a family $X \longrightarrow S$. Indeed, because $P \longrightarrow S$ is a principal bundle, for any (i, j) there is a unique $\gamma_{ij} \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma_{ij}\sigma_j = \sigma_i$ over $S_{ij} = S_i \cap S_j$; by equivariance of $P \longrightarrow \mathbb{H}$, we get

$$\gamma_{ij} f_j = f_i$$

over S_{ij}. Now, γ_{ij} defines an automorphism in $\mathcal{M}^{\mathrm{an}}$:

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{\varphi_{\gamma_{ij}}} & \mathbb{X} \\ \downarrow & \square & \downarrow \\ \mathbb{H} & \xrightarrow[\gamma_{ij}]{} & \mathbb{H} \end{array}$$

which pulls back to a morphism

$$\begin{array}{ccc} X_i|_{S_{ij}} & \xrightarrow{\varphi_{ij}} & X_j|_{S_{ij}} \\ & \searrow \quad \swarrow & \\ & S_{ij} & \end{array}$$

satisfying $\varphi_{\gamma_{ij}} \circ \varphi_j = \varphi_i \circ \varphi_{ij}$. By gluing along the maps φ_{ij} we obtain a family $X \longrightarrow S$. This defines our sought morphism $[\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}] \longrightarrow \mathcal{M}^{\mathrm{an}}$.

To prove that $[\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}] \longrightarrow \mathcal{M}^{\mathrm{an}}$ is an equivalence we exhibit a quasi-inverse, but we leave the necessary verifications to the reader. To a family of complex tori $p : X \longrightarrow S$, we can associate a locally constant sheaf

$$R_1 p_* \mathbb{Z}_X = \mathcal{H}om(R^1 p_* \mathbb{Z}_X, \mathbb{Z}_S)$$

whose stalk at each $s \in S$ is the group $H_1(X_s, \mathbb{Z})$. This comes with a symplectic $\mathbb{Z}$-bilinear pairing

$$\langle\ ,\ \rangle : R_1 p_* \mathbb{Z}_X \times R_1 p_* \mathbb{Z}_X \longrightarrow \mathbb{Z}_S$$

whose stalk at each $s \in S$ is the intersection product on $H_1(X_s, \mathbb{Z})$. We can then consider a principal $\mathrm{SL}_2(\mathbb{Z})$-bundle $P \longrightarrow S$ whose fibre at $s \in S$ is

$$P_s = \mathrm{Isom}((H_1(X_s, \mathbb{Z}), \langle\ ,\ \rangle_s), (\mathbb{Z}^{\oplus 2}, \langle\ ,\ \rangle_{\mathrm{std}})),$$

where $\langle (m,n), (m',n') \rangle_{\mathrm{std}} = mn' - m'n$ is the standard symplectic pairing. Finally, a $\mathrm{SL}_2(\mathbb{Z})$-equivariant map $P \longrightarrow \mathbb{H}$ is defined, at the fibre of $s \in S$, by

$$p \in P_s \longmapsto \frac{\int_{\gamma_2(p)} \omega}{\int_{\gamma_1(p)} \omega} \in \mathbb{H},$$

where $\gamma_1(p), \gamma_2(p)$ is the basis of $H_1(X_s, \mathbb{Z})$ corresponding to to the standard basis $(1,0)$, $(0,1)$ of $\mathbb{Z}^{\oplus 2}$ via p, and ω is an arbitrary element of $\Gamma(X_s, \Omega^1)$. □

Remark 3. It follows from the above proof that $\mathbb{H}$ is a 'fine moduli space' for pairs $(X, (\gamma_1, \gamma_2))$, where X is a complex torus and (γ_1, γ_2) is a symplectic basis of $H_1(X, \mathbb{Z})$ with respect to the intersection product. That is, $\underline{\mathbb{H}}$ is equivalent to the category over ($\mathbb{C}$-manifolds) whose objects are families of complex tori $p : X \longrightarrow S$ with a symplectic trivialisation of $R_1 p_* \mathbb{Z}_X$. The universal family over $\mathbb{H}$ (i.e., the object corresponding to $\mathrm{id}_{\mathbb{H}}$) is the complex torus $\mathbb{X} \longrightarrow \mathbb{H}$ with the symplectic trivialisation whose fibre at $\tau \in \mathbb{H}$ corresponds to the symplectic basis $(1, \tau)$ of $\mathbb{Z} + \mathbb{Z}\tau \cong H_1(\mathbb{X}_\tau, \mathbb{Z})$.

4 Hodge bundle and modular forms

We say that a diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{f} & \mathcal{B} \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle g} & \\ & \mathcal{C} & \end{array}$$

of morphisms between categories over ($\mathbb{C}$-manifolds) commutes if it is given an isomorphism of functors $h \cong g \circ f$.

Definition 4.1. A *quasi-coherent sheaf* $\mathcal{F}$ over $\mathcal{M}^{\mathrm{an}}$ is the following data.

1. For every morphism $\varphi : S \longrightarrow \mathcal{M}^{\mathrm{an}}$ corresponding to a family $X \longrightarrow S$, a quasi-coherent sheaf $\varphi^*\mathcal{F}$ on S.

2. For every commutative diagram

$$\begin{array}{ccc} S' & \xrightarrow{\;f\;} & S \\ & {\scriptstyle\varphi'}\searrow \quad \swarrow{\scriptstyle\varphi} & \\ & \mathcal{M}^{\mathrm{an}} & \end{array}$$

an isomorphism $\alpha_f : (\varphi')^*\mathcal{F} \xrightarrow{\sim} f^*\varphi^*\mathcal{F}_p$ satisfying the following cocycle relations: given a commutative diagram

$$\begin{array}{ccccc} S'' & \xrightarrow{\;f'\;} & S' & \xrightarrow{\;f\;} & S \\ & {\scriptstyle\varphi''}\searrow & \downarrow{\scriptstyle\varphi'} & \swarrow{\scriptstyle\varphi} & \\ & & \mathcal{M}^{\mathrm{an}} & & \end{array}$$

we have

$$\alpha_{f\circ f'} = \alpha_{f'} \circ (f')^*\alpha_f : (\varphi'')^*\mathcal{F} \xrightarrow{\sim} (f \circ f')^*\varphi^*\mathcal{F} = (f')^*(f^*\varphi^*\mathcal{F}).$$

A *global section* of s of $\mathcal{F}$ is a family of global sections $\varphi^*s \in \Gamma(S, \varphi^*\mathcal{F})$, for each morphism $S \longrightarrow \mathcal{M}^{\mathrm{an}}$ such that, given a morphism $f : S' \longrightarrow S$ as above, we have $\alpha_f((\varphi')^*s) = \varphi^*s$. We denote the $\mathbb{C}$-vector space of global sections by $\Gamma(\mathcal{M}^{\mathrm{an}}, \mathcal{F})$.

Modular forms are related to the following example of quasi-coherent sheaf.

Example 8 (Hodge bundle)**.** Given a morphism $\varphi : S \longrightarrow \mathcal{M}^{\mathrm{an}}$ corresponding to a family of complex tori $p : X \longrightarrow S$, we set $\varphi^*\mathcal{F} := p_*\Omega^1_{X/S}$. It follows the 'cohomology and base change theorems' (see [1, II.5]) that $\varphi^*\mathcal{F}$ is a line bundle (invertible sheaf) over S whose fibre at each $s \in S$ is the space of global holomorphic 1-forms $\Gamma(X_s, \Omega^1)$, and that the formation of $\varphi^*\mathcal{F}$ commutes with every base change in S. This defines a quasi-coherent sheaf $\mathcal{F}$ on $\mathcal{M}^{\mathrm{an}}$.

Let $k \in \mathbb{Z}$ and consider a global section $s \in \Gamma(\mathcal{M}^{\mathrm{an}}, \mathcal{F}^{\otimes k})$. It gives in particular a global section

$$\pi^*s \in \Gamma(\mathbb{H}, \pi^*\mathcal{F}^{\otimes k})$$

corresponding to the family $\mathbb{X} \longrightarrow \mathbb{H}$ of Example 5. The line bundle $\pi^*\mathcal{F}$ over $\mathbb{H}$ can be trivialized by a global section

$$\omega \in \Gamma(\mathbb{H}, \pi^*\mathcal{F})$$

whose fibre at each $\tau \in \mathbb{H}$ is the 1-form $\omega(\tau) = 2\pi i\, dz$ on the complex torus $\mathbb{X}_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$. Thus there exists a unique holomorphic function $f : \mathbb{H} \longrightarrow \mathbb{C}$ such that

$$\pi^*s = f\omega^{\otimes k}.$$

Now, since π^*s comes from a global section over $\mathcal{M}^{\mathrm{an}}$, the function f is not arbitrary. For every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, we have an automorphism

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{\;\varphi_\gamma\;} & \mathbb{X} \\ \big\downarrow & \square & \big\downarrow \\ \mathbb{H} & \xrightarrow[\;\gamma\;]{} & \mathbb{H} \end{array}$$

in the category $\mathcal{M}^{\mathrm{an}}$, where $\varphi_{\gamma,\tau} : \mathbb{X}_\tau \longrightarrow \mathbb{X}_{\gamma\tau}$ is given by multiplication by $(c\tau+d)^{-1}$ (see Example 4). Since we must have $\gamma^*\pi^*s = \pi^*s$ under the natural identifications, and since $\gamma^*\omega = (c\tau+d)^{-1}\omega$, we conclude that

$$f(\gamma\tau)(c\tau+d)^{-k}\omega^{\otimes k} = f(\tau)\omega^{\otimes k}.$$

Thus f satisfies

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}). \tag{2}$$

We showed that to any global section $s \in \Gamma(\mathcal{M}^{\mathrm{an}}, \mathcal{F}^{\otimes k})$ we can associate a holomorphic function $f : \mathbb{H} \longrightarrow \mathbb{C}$ satisfying the modularity property (2).

Theorem 4.2. *If $f : \mathbb{H} \longrightarrow \mathbb{C}$ is a holomorphic function satisfying (2), then there exists a unique global section $s \in \Gamma(\mathcal{M}^{\mathrm{an}}, \mathcal{F}^{\otimes k})$ such that $\pi^*s = f\omega^{\otimes k}$.*

Proof. For the existence, let $\varphi : S \longrightarrow \mathcal{M}^{\mathrm{an}}$ be a morphism corresponding to a family of complex tori $p : X \longrightarrow S$. Consider the corresponding object $(P \longrightarrow S, P \longrightarrow \mathbb{H})$ corresponding to $p : X \longrightarrow S$ via Theorem 3.4. Since $P \longrightarrow S$ is locally trivial, we can cover S by open subsets S_i with sections of $P \longrightarrow S$, whose composition with the equivariant map $P \longrightarrow \mathbb{H}$ gives maps $f_i : S_i \longrightarrow \mathbb{H}$. We set $s_i = f_i^*(f\omega^{\otimes k})$ and we use the modularity of f to check that s_i glue to a section $\varphi^*s \in \Gamma(S, \varphi^*\mathcal{F})$.

For the unicity, we use again the fact that, locally over S, every $\varphi : S \longrightarrow \mathcal{M}^{\mathrm{an}}$ factors through $\pi : \mathbb{H} \longrightarrow \mathcal{M}^{\mathrm{an}}$ (Theorem 3.4). □

To give a geometric interpretation of the condition at infinity in the defintion of a modular form, we must compactify the moduli stack $\mathcal{M}^{\mathrm{an}}$. For simplicity, let us just mention that a 'neighborhood at infinity' in the compactification $\overline{\mathcal{M}}^{\mathrm{an}}$ is given by the Tate family.

Example 9 (Tate family of complex tori)**.** Let $\mathbb{Z}$ act on $\mathbb{C}^* \times D^*$ by $n \cdot (t, q) = (tq^n, q)$. This action is proper and free, and the quotient gives a 2-dimensional complex manifold $\mathbb{T}$ with a proper holomorphic map $\mathbb{T} \longrightarrow D^*$. In fact, this is a family of complex tori whose fibre at $q \in D^*$ is $\mathbb{T}_q = \mathbb{C}^*/q^{\mathbb{Z}}$. Moreover, there is a morphism in $\mathcal{M}^{\mathrm{an}}$

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{\varphi} & \mathbb{T} \\ \downarrow & \square & \downarrow \\ \mathbb{H} & \xrightarrow[q]{} & D^* \end{array}$$

where $q(\tau) = e^{2\pi i\tau}$ and $\varphi_\tau : \mathbb{X}_\tau \xrightarrow{\sim} \mathbb{T}_{q(\tau)}$ is given by $z \longmapsto e^{2\pi iz}$.

Note that the 1-form dt/t on $\mathbb{C}^*$ descends to a non-zero element of $\Gamma(\mathbb{T}_{q(\tau)}, \Omega^1)$ which pulls back to $\omega(\tau) = 2\pi idz$ on $\mathbb{X}_\tau$ via the above isomorphism. Let

$$\pi_{\mathbb{T}} : D^* \longrightarrow \mathcal{M}^{\mathrm{an}}$$

be the morphism corresponding to the Tate family and $\omega_{\mathbb{T}}$ be the trivialisation of $\pi_{\mathbb{T}}^*\mathcal{F}$ whose fibre at each $q \in D^*$ is the 1-form $\omega_{\mathbb{T}}(q) = dt/t$ on $\mathbb{T}_q$. If $s \in \Gamma(\mathcal{M}^{\mathrm{an}}, \mathcal{F}^{\otimes k})$ corresponds to $f : \mathbb{H} \longrightarrow \mathbb{C}$ via $\pi^*s = f\omega^{\otimes k}$, then

$$\pi_{\mathbb{T}}^*s = (\sum_{n\in\mathbb{Z}} a_n q^n)\omega_{\mathbb{T}}^{\otimes k}$$

where

$$f(\tau) = \sum_{n\in\mathbb{Z}} a_n q^n, \qquad q = e^{2\pi i\tau}.$$

We say that s is *holomorphic at infinity* if $a_n = 0$ for every $n < 0$. We conclude that a modular form of weight k is a global section of $\mathcal{F}^{\otimes k}$ over $\mathcal{M}^{\mathrm{an}}$ which is holomorphic at infinity.

Remark 4. Given a suitable definition of the compactification $\overline{\mathcal{M}}^{\mathrm{an}}$, one can show that the Hodge bundle $\mathcal{F}$ extends to a quasi-coherent sheaf $\overline{\mathcal{F}}$ on $\overline{\mathcal{M}}^{\mathrm{an}}$, and that a section $s \in \Gamma(\mathcal{M}^{\mathrm{an}}, \mathcal{F}^{\otimes k})$ is holomorphic at infinity if and only if it extends to $\Gamma(\overline{\mathcal{M}}^{\mathrm{an}}, \overline{\mathcal{F}}^{\otimes k})$.

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