A crash course in modular forms and cohomology - Lecture 2

Tiago J. Fonseca *

Mathematical Institute, University of Oxford

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1 Elliptic curves

By a *curve* over a field k, we mean a 1-dimensional, geometrically integral and separated scheme of finite type over k.

Definition 1.1. An *elliptic curve* over k is a genus 1 proper smooth curve E over k with a rational point $O \in E(k)$.

We recall that 'genus 1' means that $\dim \Gamma(E, \Omega^1_{E/k}) = 1$.

Example 1 (Classical Weierstrass form)**.** Suppose that k is of characteristic $\neq 2, 3$, and consider the projective plane curve $E_{(u,v)} \subset \mathbb{P}^2_k$ defined by

$$y^2 z = 4x^3 - uxz^2 - vz^3, \qquad u, v \in k,\ u^3 - 27v^2 \neq 0$$

with the point $O = (0 : 1 : 0) \in E_{(u,v)}(k)$ ('point at infinity'). Note that $E_{(u,v)} \setminus \{O\} \subset \mathbb{A}^2_k$ is the affine plane curve given by the equation $y^2 = 4x^3 - ux - v$. The condition $u^3 - 27v^2 \neq 0$ ensures that $E_{(u,v)}$ is smooth. Further, one can show that the differential form

$$\omega = \frac{dx}{y}$$

defined on the open subset of $E_{(u,v)} \setminus \{O\}$ where $y \neq 0$ extends to a global section of $\Omega^1_{E_{(u,v)}/k}$ and generates this sheaf as an $\mathcal{O}_{E_{(u,v)}}$-module ([5], Proposition 1.5), so that

$$\Gamma(E_{(u,v)}, \Omega^1) = \Gamma(E, \mathcal{O})\omega = k\omega.$$

*tiago.jardimdafonseca@maths.ox.ac.uk

Every elliptic curve E admits the structure of a (commutative) group scheme over k for which O is the identity. Briefly, it is induced from the group law on the Picard group of E via the isomorphism

$$E \xrightarrow{\sim} \mathrm{Pic}^0_{E/k}, \qquad P \longmapsto \mathcal{O}_E([P]-[O]).$$

More concretely, one can show via Riemann-Roch that $\mathcal{O}_E(3[O])$ is very ample, and deduce from there that, if $\mathrm{char}(k) \neq 2,3$, then E admits a Weierstrass equation as in Example 1. The group law on E is then determined by the geometric conditions:

- The opposite $-P$ of a point P is its reflexion about the x-axis
- Three points P_1, P_2, P_3 satisfy $P_1+P_2+P_3=O$ if and only if they are colinear in $\mathbb{P}^2_k$.

We refer to [5], Section III.2, for details on the group structure.

The above discussion shows in particular that, if E is an elliptic curve over $\mathbb{C}$, then the corresponding complex manifold $E(\mathbb{C})$ is a complex torus, as defined in the last lecture. Conversely, every complex torus is the analytification of an elliptic curve; this follows from the next theorem of Weierstrass, which also gives an explicit formula for $\exp : \mathrm{Lie}\, E \longrightarrow E(\mathbb{C})$.

Theorem 1.2 (Weierstrass)**.** *Let $\Lambda \subset \mathbb{C}$ be a lattice.*

1. *Set*

$$g_2(\Lambda) = 60 \sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{\lambda^4}, \qquad g_3(\Lambda) = 140 \sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{\lambda^6}.$$

Then $g_2(\Lambda)^3 - 27 g_3(\Lambda)^2 \neq 0$, and the meromorphic function

$$\wp_\Lambda(z) = \frac{1}{z^2} - \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right)$$

satisfies the differential equation

$$\wp'_\Lambda(z)^2 = 4\wp_\Lambda(z)^3 - g_2(\Lambda)\wp_\Lambda(z) - g_3(\Lambda).$$

2. *We have an isomorphism of complex tori*

$$\mathbb{C}/\Lambda \xrightarrow{\sim} E_{(g_2(\Lambda), g_3(\Lambda))}(\mathbb{C}), \qquad z \longmapsto \begin{cases} (\wp_\Lambda(z) : \wp'_\Lambda(z) : 1) & \text{if } z \neq 0 \\ (0:1:0) & \text{if } z = 0 \end{cases}$$

3. *Let $u, v \in \mathbb{C}$ with $u^2 - 27v^2 \neq 0$. Then*

$$\Lambda = \{ \int_\gamma \frac{dx}{y} \mid \gamma \in H_1(E_{(u,v)}(\mathbb{C}), \mathbb{Z}) \}$$

is a lattice such that $g_2(\Lambda) = u$ and $g_3(\Lambda) = v$.

Proof. See [5], Chapter VI. □

Note that, given $\tau \in \mathbb{H}$, we have

$$g_2(\mathbb{Z} + \mathbb{Z}\tau) = 60 G_4(\tau) \qquad \text{and} \qquad g_3(\mathbb{Z} + \mathbb{Z}\tau) = 140 G_6(\tau)$$

where G_k is the Eisenstein series introduced in the last lecture.

2 The moduli stack of elliptic curves

Our next goal is to 'algebraise' the moduli stack of complex tori by considering a moduli stack of elliptic curves. For this, we need the concept of families.

Definition 2.1. Let S be a scheme. A *family of elliptic curves over* S, or simply an *elliptic curve over* S, is a proper smooth morphism $p : E \longrightarrow S$ with a section $O : S \longrightarrow E$ whose geometric fibres are elliptic curves.

The group law on the fibres extends to a structure of S-group scheme on $p : E \longrightarrow S$ (see [4], Section 2.1).

Example 2 (Weierstrass family)**.** If $S = \operatorname{Spec} \mathbb{Z}[1/6, u, v, (u^3 - 27v^2)^{-1}]$, then there is family of elliptic curves over S the subscheme $E \subset \mathbb{P}^2_{\mathbb{Z}[1/6]} \times_{\mathbb{Z}[1/6]} S$ defined by the equation $y^2z = 4x^3 - uxz^2 - vz^3$. The geometric fibres of $\mathrm{pr}_2 : E \longrightarrow S$ are the elliptic curves of Example 1.

In what follows, for simplicity, we shall only consider $\mathbb{Q}$-schemes.

Definition 2.2 (Moduli stack of elliptic curves)**.** We define a category $\mathcal{M}$ as follows. Its objects are given by families of elliptic curves $p : E \longrightarrow S$ over arbitrary $\mathbb{Q}$-schemes, and its morphisms are given by Cartesian squares

$$\begin{array}{ccc} E' & \longrightarrow & E \\ \downarrow & \square & \downarrow \\ S' & \longrightarrow & S \end{array}$$

The category $\mathcal{M}$ comes with a natural forgetful functor to the category of $\mathbb{Q}$-schemes

$$\mathcal{M} \longrightarrow (\mathbb{Q}\text{-schemes}), \qquad (p : E \longrightarrow S) \longmapsto S$$

giving it the structure of a *category over the category of* $\mathbb{Q}$*-schemes*. If R is a $\mathbb{Q}$-algebra, we denote by $\mathcal{M}_R \longrightarrow (R\text{-schemes})$ the analogous category where we replace $\mathbb{Q}$-schemes by R-schemes.

Remark 1. Classically, $\mathcal{M}$ is denoted by $\mathcal{M}_{1,1}$ because it classifies genus 1 curves with 1 marked point. Other notations encountered in the literature are $\mathcal{A}_1$ and (Ell).

The moduli stack of complex tori $\mathcal{M}^{\mathrm{an}}$ should be thought as the analytification of $\mathcal{M}_{\mathbb{C}}$, since the analytification of a family of elliptic curves $p : E \longrightarrow S$, with S a smooth scheme over $\mathbb{C}$, is a family of complex tori $p^{\mathrm{an}} : E(\mathbb{C}) \longrightarrow S(\mathbb{C})$, as defined in the last lecture. Note however that not every complex manifold is the analytification of an algebraic variety (e.g. $\mathbb{H}$), so not every family of complex tori is the analytification of a family of elliptic curves.

Recall that $\mathcal{M}^{\mathrm{an}}$ admits simple description as a stacky quotient of $\mathbb{H}$ by the action of the discrete group $\mathrm{SL}_2(\mathbb{Z})$. We saw that $\mathbb{H}$ is in fact the moduli space of complex tori X equipped with an extra structure, namely a symplectic basis of $H_1(X, \mathbb{Z})$, and that the action of $\mathrm{SL}_2(\mathbb{Z})$ is the action on such bases. A key property here is that this extra structure *rigidifies* the moduli problem of complex tori: the only automorphism of X preserving a symplectic basis of $H_1(X, \mathbb{Z})$ is the identity. To obtain a similar description for $\mathcal{M}$, we need to rigidify the moduli problem of elliptic curves with some algebraic extra structure.

Definition 2.3. Let $E \longrightarrow S$ be a family of elliptic curves and $n \geq 1$ be an integer. A *full level* n *structure* on $E \longrightarrow S$ is an isomorphism of S-group schemes

$$E[n] \xrightarrow{\sim} (\mathbb{Z}/n\mathbb{Z})_S^{\oplus 2},$$

where $E[n]$ denotes the n-torsion S-subscheme of E.

Note that full level n structures are compatible with pullbacks, so we can form a category $\mathcal{M}_n$ over ($\mathbb{Q}$-schemes) whose objects are pairs $(E \longrightarrow S, \alpha)$, where $E \longrightarrow S$ is a family of elliptic curves and α is a full level n structure on $E \longrightarrow S$, and morphisms are Cartesian squares of families of elliptic curves preserving the full level n-structures.

There is a forgetful functor $\mathcal{M}_n \longrightarrow \mathcal{M}$ and the group $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$ acts on $\mathcal{M}_n$ over $\mathcal{M}$: every $g \in \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$ yields a functor

$$\mathcal{M}_n \longrightarrow \mathcal{M}_n, \qquad (E \longrightarrow S, \alpha) \longmapsto (E \longrightarrow S, g\alpha)$$

preserving the forgetful functor $\mathcal{M}_n \longrightarrow \mathcal{M}$.

Theorem 2.4. *For every integer $n \geq 3$, there exists a smooth affine curve M_n over $\mathbb{Q}$ with an equivalence of categories over ($\mathbb{Q}$-schemes)*

$$M_n \rightleftarrows \mathcal{M}_n.$$

Moreover, the natural morphism $M_n \longrightarrow \mathcal{M}$ factors into an equivalence of categories

$$[\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})\backslash M_n] \rightleftarrows \mathcal{M}.$$

Sketch of proof. Let t, s be indeterminates, $a_1 = 3t - 1$, $a_3 = -3t^2 - s - 3ts$, and

$$M_3 = \operatorname{Spec} \mathbb{Q}[t, s^{\pm 1}, \Delta^{-1}]/(t^3 - (t+s)^3), \qquad \Delta = (a_1^3 - 27a_3)a_3.$$

Over M_3, we can define a family of elliptic curves $E \longrightarrow M_3$ by the homogeneous equation

$$y^2 z + a_1 xyz + a_3 yz^2 = x^3 \tag{1}$$

and $O = (0 : 1 : 0)$. Further, $E \longrightarrow M_3$ admits a full level 3 structure α corresponding to the sections

$$P = (0 : 0 : 1), \qquad Q = (s : t + s : 1)$$

of $E[3] \longrightarrow M_3$. We thus obtain an object $(E \longrightarrow M_3, \alpha)$ of $\mathcal{M}_3$, which corresponds to a morphism $M_3 \longrightarrow \mathcal{M}_3$. To show that this is an equivalence of categories, we start with an object $(E \longrightarrow S, \alpha)$ of $\mathcal{M}_3$, put it in 'long Weierstrass form' (locally over S), and observe that the nine 3 torsion points correspond to the nine inflexion points of this cubic. This allows to construct a change of variables in which the equation for E has the form (1) above. We refer to [4] (2.2.10) for details. The general case actually follows from this one by a standard yoga, because $\mathcal{M}_n \longrightarrow \mathcal{M}$ is 'relatively representable' and 'rigid' for $n \geq 3$ (see [4] Chapter 4). □

Remark 2. The curves M_n are examples of *modular curves.* They give finite étale coverings of $\mathcal{M}$, in the sense of stacks. This shows that $\mathcal{M}$ is a *Deligne-Mumford stack* (see [6]).

Let us briefly explain how to relate M_n with the uniformisation $\pi : \mathbb{H} \longrightarrow \mathcal{M}^{\mathrm{an}}$.

Given an object $E \longrightarrow S$ of $\mathcal{M}$, and $n \geq 1$ an integer, there's a canonical pairing

$$e_n : E[n] \times_S E[n] \longrightarrow \mu_{n,S}$$

where μ_n is the group of nth roots of unit; this is the *Weil pairing* (see [5] III.8 and [4] 2.8). A *symplectic full level n structure* on an object $E \longrightarrow S$ of $\mathcal{M}_{\mathbb{C}}$ is a full level n structure $\alpha = (P, Q)$ such that $e_n(P, Q) = e^{\frac{2\pi i}{n}}$.

Over complex tori, the Weil pairing is particularly simple. If X is a complex torus, then the n torsion $X[n]$ can be identified with

$$\frac{1}{n} H_1(X, \mathbb{Z})/H_1(X, \mathbb{Z}) = H_1(X, \mathbb{Z})/nH_1(X, \mathbb{Z})$$

and the Weil pairing is then simply given by the reduction modulo n of the intersection form on $H_1(X,\mathbb{Z})$, after the identification $\mathbb{Z}/n\mathbb{Z} \cong \mu_n$ given by $1 \longmapsto e^{\frac{2\pi i}{n}}$. In particular, every symplectic basis of $H_1(X,\mathbb{Z})$ induces a symplectic full level n structure on X. If $\Gamma(n) \leq \mathrm{SL}_2(\mathbb{Z})$ denotes the subgroup of matrices reducing to the identity module n, then

$$Y(n) := \Gamma(n)\backslash\mathbb{H}$$

classifies complex tori with a symplectic full level n-structure.

It follows from the above discussion that $M_n(\mathbb{C})$ breaks up as a disjoint union

$$M_n(\mathbb{C}) = \bigsqcup_{\zeta} M_n(\mathbb{C})_\zeta, \qquad M_n(\mathbb{C})_\zeta \cong Y(n)$$

where ζ runs through the set of primitive nth roots of unity. Here, the connected component $M_n(\mathbb{C})_\zeta$ corresponds to the full level n structures $\alpha = (P,Q)$ such that $w(P,Q) = \zeta$.

Remark 3. Some of the spaces $Y(n)$ admit a classical description. For instance, $Y(3)$ can be identified with $\operatorname{Spec}\mathbb{C}[t,(t^3-1)^{-1}]$ with universal family of elliptic curves given by the Hesse pencil $x^3+y^3+z^3 = 3txyz$ (the identity section being given by $O = (1:-1:0)$). The modular curve $Y(7)$ can be identified with an open subset of the Klein quartic $x^3y+y^3z+z^3x = 0$ in $\mathbb{P}^2$.[1] Let us also remark that $Y(n)$ is algebraic, and can actually be defined over $\mathbb{Q}(\mu_n)$.

3 Modular forms and the Hodge bundle, the algebraic picture

We define quasi-coherent sheaves over $\mathcal{M}$ in the same way we did for $\mathcal{M}^{\mathrm{an}}$. The *Hodge bundle* is the quasi-coherent sheaf $\mathcal{F}$ over $\mathcal{M}$ given by

$$\varphi^*\mathcal{F} = p_*\Omega^1_{E/S}$$

for a morphism $\varphi : S \longrightarrow \mathcal{M}$ corresponding to a family of elliptic curves $p : E \longrightarrow S$. Similarly, for any $\mathbb{Q}$-algebra R, we have a Hodge bundle $\mathcal{F}$ over $\mathcal{M}_R$, and we have

$$\Gamma(\mathcal{M}_R,\mathcal{F}^{\otimes k}) = \Gamma(\mathcal{M},\mathcal{F}^{\otimes k}) \otimes_{\mathbb{Q}} R$$

by flat base change.

To understand the relation between the algebraic $\Gamma(\mathcal{M}_{\mathbb{C}},\mathcal{F}^{\otimes k})$ and the analytic $\Gamma(\mathcal{M}^{\mathrm{an}},\mathcal{F}^{\otimes k})$, note that if S is a smooth $\mathbb{C}$-scheme, and $\varphi : S \longrightarrow \mathcal{M}_{\mathbb{C}}$ is a morphism corresponding to a family of elliptic curves $p : E \longrightarrow S$, then we have a morphism $\varphi^{\mathrm{an}} : S(\mathbb{C}) \longrightarrow \mathcal{M}^{\mathrm{an}}$ corresponding to the family of complex tori $p^{\mathrm{an}} : E(\mathbb{C}) \longrightarrow S(\mathbb{C})$.

Lemma 3.1. *There is a unique $\mathbb{C}$-linear map*

$$\Gamma(\mathcal{M}_{\mathbb{C}},\mathcal{F}^{\otimes k}) \longrightarrow \Gamma(\mathcal{M}^{\mathrm{an}},\mathcal{F}^{\otimes k}), \qquad s \longmapsto s^{\mathrm{an}}$$

such that

$$(\varphi^{\mathrm{an}})^* s^{\mathrm{an}} = (\varphi^* s)^{\mathrm{an}}$$

for every morphism $\varphi : S \longrightarrow \mathcal{M}$, where S is a smooth $\mathbb{C}$-scheme.

[1] See N. D. Elkies, *The Klein quartic in number theory.* In The EightfoldWay: The Beauty of Klein's Quartic Curve, ed. Sylvio Levi, 51-102. Mathematical Sciences Research Institute publications, 35. Cambridge University Press.

Here, if S is a smooth $\mathbb{C}$-scheme, $\mathcal{E}$ is a quasi-coherent sheaf on S, and $e \in \Gamma(S, \mathcal{E})$ is a global section, we denote by $e^{\mathrm{an}} \in \Gamma(S(\mathbb{C}), \mathcal{E}^{\mathrm{an}})$ the corresponding global section of the analytic quasi-coherent sheaf $\mathcal{E}^{\mathrm{an}}$ on the complex manifold $S(\mathbb{C})$.

Proof. Let $\pi : \mathbb{H} \longrightarrow \mathcal{M}^{\mathrm{an}}$ be the uniformisation. Since $\mathcal{M}^{\mathrm{an}}$ is equivalent to $[\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}]$, it suffices to define $\pi^* s^{\mathrm{an}}$ such that $\gamma^*\pi^* s^{\mathrm{an}} = \pi^* s^{\mathrm{an}}$ for every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Given $n \geq 3$, the action of $\Gamma(n)$ on $\mathbb{H}$ is free, and the family of complex tori $\mathbb{X} \longrightarrow \mathbb{H}$ descends to a family of complex tori $X \longrightarrow Y(n)$:

$$\begin{array}{ccc} \mathbb{X} & \longrightarrow & X \\ \big\downarrow & \square & \big\downarrow \\ \mathbb{H} & \longrightarrow & Y(n) \end{array}$$

Now, it follows from last section that $Y(n)$ is algebraic and that $X \longrightarrow Y(n)$ is the analytification of a family of elliptic curves. This allows us to define $\pi^* s^{\mathrm{an}}$ by pullback. $\square$

The map $\Gamma(\mathcal{M}_{\mathbb{C}}, \mathcal{F}^{\otimes k}) \longrightarrow \Gamma(\mathcal{M}^{\mathrm{an}}, \mathcal{F}^{\otimes k})$ is injective, but is not surjective in general. In other words, not every global section $s \in \Gamma(\mathcal{M}^{\mathrm{an}}, \mathcal{F}^{\otimes k})$ is algebraic. The next result shows that, if s comes from a modular form, so that it also satisfies a condition 'at infinity', then s is actually algebraic.

Theorem 3.2. *Let f be a modular form of weight k. Then there is a unique $s \in \Gamma(\mathcal{M}_{\mathbb{C}}, \mathcal{F}^{\otimes k})$ such that $\pi^* s^{\mathrm{an}} = f\omega^{\otimes k}$.*

Sketch of the proof. Let $s' \in \Gamma(\mathcal{M}^{\mathrm{an}}, \mathcal{F}^{\otimes k})$ be the unique global section such that $\pi^* s' = f\omega^{\otimes k}$, let $n \geq 3$, and denote by $p_n : M_{n,\mathbb{C}} \longrightarrow \mathcal{M}_{\mathbb{C}}$ the morphism given by Theorem 2.4.

The pullback $(p_n^{\mathrm{an}})^* s'$ is an analytic section over the smooth curve $M_n(\mathbb{C})$, and our goal is to prove that it extends to the compactification $\overline{M}_n(\mathbb{C})$. For this we use that the coordinates at the 'points at infinity' are given by gluing the unit disc D with $M_n(\mathbb{C})$ along the maps

$$\varphi_{n,[\alpha]} : D^* \longrightarrow M_n(\mathbb{C})$$

corresponding to $(T_n \longrightarrow D^*, \alpha)$, where $T_n \longrightarrow D^*$ is the 'level n Tate family', obtained by pullback

$$\begin{array}{ccc} T_n & \longrightarrow & T \\ \big\downarrow & \square & \big\downarrow \\ D^* & \xrightarrow[q \mapsto q^n]{} & D^* \end{array}$$

and $[\alpha]$ is an isomorphism class of a full level n structure α on $T_n \longrightarrow D^*$ (see [3] Section 1.4, [4] Chapters 8-10, and [1] Section 2.4). Moreover, the Hodge line bundle $(p_n^{\mathrm{an}})^*\mathcal{F}$ over $M_n(\mathbb{C})$ extends to a line bundle $(p_n^{\mathrm{an}})^*\overline{\mathcal{F}}$ over the compactification $\overline{M}_n(\mathbb{C})$ via the trivialisation dt/t of $\pi_{T_n}^*\mathcal{F}$ (see last lecture). Now, it follows from the fact that f is bounded at infinity, and from the commutative diagrams

$$\begin{array}{ccc} D^* & \xrightarrow{q \mapsto q^n} & D^* \\ {\scriptstyle \varphi_{n,[\alpha]}}\big\downarrow & \searrow^{\pi_{T_n}} & \big\downarrow{\scriptstyle \pi_T} \\ M_n(\mathbb{C}) & \xrightarrow[p_n^{\mathrm{an}}]{} & \mathcal{M}^{\mathrm{an}} \end{array}$$

that $(p_n^{\mathrm{an}})^* s'$ extends to a global section of $(p_n^{\mathrm{an}})^*\overline{\mathcal{F}}^{\otimes k}$ over $\overline{M}_n(\mathbb{C})$.

Since $M_{n,\mathbb{C}}$ is proper, it follows from Serre's GAGA that $(p_n^{\mathrm{an}})^* s'$ is algebraic, i.e., $(p_n^{\mathrm{an}})^* s' = t^{\mathrm{an}}$ for some $t \in \Gamma(M_{n,\mathbb{C}}, p_n^*\mathcal{F}^{\otimes k})$. Since $(p_n^{\mathrm{an}})^* s'$ comes from a global section $s' \in \Gamma(\mathcal{M}^{\mathrm{an}}, \mathcal{F}^{\otimes k})$, it is stable under the action of $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$, and the same is true for t. This shows that $t = p_n^* s$ for some $s \in \Gamma(\mathcal{M}_{\mathbb{C}}, \mathcal{F}^{\otimes k})$ and that $s' = s^{\mathrm{an}}$. $\square$

We can thus reformulate the concept of a modular form in a completely algebraic way. For this, let us consider the following algebraic analog of the Tate family.

Example 3 (Tate curve)**.** The Tate curve is the elliptic curve $\hat{T}$ over $\mathbb{Q}((q)) = \mathbb{Q}[[q]][q^{-1}]$ defined by the Weierstrass equation

$$y^2 z = 4x^3 - \frac{E_4(q)}{12} xz^2 + \frac{E_6(q)}{216} z^3$$

where

$$E_4(q) = 1 + 240 \sum_{n \geq 1} \sigma_3(n) q^n \qquad \text{and} \qquad E_6(q) = 1 - 504 \sum_{n \geq 1} \sigma_5(n) q^n.$$

This is related to the Tate family of complex tori $T \longrightarrow D^*$ via the Weierstrass isomorphism: for every $q \in D^*$, we have

$$\begin{aligned} \mathbb{C}^\times / q^{\mathbb{Z}} &\xrightarrow{\sim} E_{(E_4(q)/12, -E_6(q)/216)}(\mathbb{C}) \\ t &\longmapsto \begin{cases} ((\frac{1}{2\pi i})^2 \wp_{\frac{\log q}{2\pi i}}(\frac{\log t}{2\pi i}) : (\frac{1}{2\pi i})^3 \wp'_{\frac{\log q}{2\pi i}}(\frac{\log t}{2\pi i}) : 1) & \text{if } t \neq 1 \\ (0 : 1 : 0) & \text{if } t = 1 \end{cases} \end{aligned}$$

Thus, $\hat{T}_{\mathbb{C}} \longrightarrow \operatorname{Spec} \mathbb{Q}((q)) \otimes \mathbb{C}$ is the 'formal completion' of $T \longrightarrow D^*$ at 0.

The pullback of a section $s \in \Gamma(\mathcal{M}_R, \mathcal{F}^{\otimes k})$ via $\pi_{\hat{T}_R} : \operatorname{Spec} \mathbb{Q}((q)) \otimes R \longrightarrow \mathcal{M}_R$ corresponding to the Tate elliptic curve is of the form

$$\pi^*_{\hat{T}_R} s = f(q) \omega^{\otimes k}_{\hat{T}_R}, \qquad f(q) \in \mathbb{Q}((q)) \otimes R$$

where $\omega_{\hat{T}_R}$ is the trivialisation of the Hodge bundle of $\hat{T}_R \longrightarrow \operatorname{Spec} \mathbb{Q}((q)) \otimes R$ given by dx/y. If $f(q) \in \mathbb{Q}[[q]] \otimes R$, we say that s is *holomorphic at infinity.* We denote the R-submodule of such sections by $\Gamma(\overline{\mathcal{M}}_R, \overline{\mathcal{F}}^{\otimes k})$.

Definition 3.3. Let k be an integer and R be a $\mathbb{Q}$-algebra. A *weakly holomorphic modular form of weight k over R* is an element of $\Gamma(\mathcal{M}_R, \mathcal{F}^{\otimes k})$. A *modular form of weight k over R* is an element of $\Gamma(\overline{\mathcal{M}}_R, \overline{\mathcal{F}}^{\otimes k})$.

We have seen that a modular form of weight k over $\mathbb{C}$ is the same thing as a (holomorphic) modular form of weight k. Weakly holomorphic modular forms correspond to holomorphic functions $f : \mathbb{H} \longrightarrow \mathbb{C}$ satisfying a modularity condition of weight k, and which are 'meromorphic at infinity', meaning that its Fourier expansion $f = \sum_{n \in \mathbb{Z}} a_n q^n$ is a finite tailed Laurent series in q.

If R is a subalgebra of $\mathbb{C}$, then a modular form of weight k over R gives rise to a modular form $f : \mathbb{H} \longrightarrow \mathbb{C}$ of weight k whose Fourier coefficients a_n lie in R. Conversely, we have the following result, the proof of which we refer to [3] Section 1.6.

Theorem 3.4 (q-expansion principle)**.** *If $f : \mathbb{H} \longrightarrow \mathbb{C}$ is a (weakly holomorphic) modular form of weight k whose Fourier coefficients all lie in some subalgebra R of $\mathbb{C}$, then f comes from an element of $\Gamma(\mathcal{M}_R, \mathcal{F}^{\otimes k})$.*

4 De Rham cohomology and the Gauss-Manin connection

Definition 4.1. Let X be a scheme over a field k. For every $n \geq 0$, the nth *algebraic de Rham cohomology of* X *over* k is the nth hypercohomology group of the complex of differential forms $\Omega^\bullet_{X/k}$:

$$H^n_{\mathrm{dR}}(X/k) := \mathbb{H}^n(X, \Omega^\bullet_{X/k}).$$

By abuse, we simply write $H^n_{\mathrm{dR}}(X)$ if k is understood.

If $X = \operatorname{Spec} A$ is affine, it follows from Serre's vanishing theorem that the algebraic de Rham cohomology can be computed by the cohomology of the complex of global sections, so that

$$H^n_{\mathrm{dR}}(X) = \frac{\ker(d : \Omega^n_{A/k} \longrightarrow \Omega^{n+1}_{A/k})}{\operatorname{im}(d : \Omega^{n-1}_{A/k} \longrightarrow \Omega^n_{A/k})} = \frac{\text{closed forms}}{\text{exact forms}}.$$

Example 4. Let k be of characteristic 0 and $X = \mathbb{G}_m = \operatorname{Spec} k[x, x^{-1}]$. The de Rham complex is given by

$$\begin{gathered} 0 \longrightarrow k[x, x^{-1}] \xrightarrow{d} k[x, x^{-1}]dx \longrightarrow 0 \\ \sum_i a_i x^i \longmapsto (\sum_i i a_i x^{i-1})dx \end{gathered}$$

Every $x^m dx$ is in the image of d, except for $m = -1$. This proves that

$$H^1_{\mathrm{dR}}(\mathbb{G}_m) = \operatorname{coker}(k[x, x^{-1}] \xrightarrow{d} k[x, x^{-1}]dx) = k \cdot [dx/x].$$

We are interested in the cohomology of an elliptic curve. To compute it, we will use the following general fact.

Lemma 4.2. *Let C be a smooth projective curve over a field k of characteristic 0. If $P \in C(k)$ is a rational point, then there is a natural isomorphism $H^1_{\mathrm{dR}}(C) \xrightarrow{\sim} H^1_{\mathrm{dR}}(C \setminus \{P\})$.* □

Example 5 (De Rham cohomology of an elliptic curve)**.** Let k be a field of characteristic 0 and $E = E_{(u,v)}$ be an elliptic curve over k with (affine) Weierstrass equation $y^2 = f(x)$, where $f(x) = 4x^3 - ux - v$ is such that $u^3 - 27v^2 \neq 0$. It follows from the above lemma that $H^1_{\mathrm{dR}}(E) = H^1_{\mathrm{dR}}(E \setminus \{O\})$, where

$$E \setminus \{O\} = \operatorname{Spec} A, \qquad A = k[x, y]/(y^2 - f(x)).$$

Recall that $\Omega^1_{E/k}$ is trivialised by $\omega = dx/y = 2dy/f'(x)$, so that the de Rham complex of $E \setminus \{O\}$ over k is given by

$$0 \longrightarrow A \xrightarrow{d} A\omega \longrightarrow 0.$$

To compute d, note that every element h of A can be written uniquely as $h = P + Qy$, with $P, Q \in k[x]$. Thus

$$dh = P'(x)dx + Q'(x)ydx = \left(\left(Q'f + \frac{1}{2}Qf'\right) + P'y\right)\omega.$$

If Q has leading term $a_d x^d$, then $Q'f + \frac{1}{2}Qf'$ has leading term $(4d+6)a_d x^{d+2}$ (note that $4d+6$ is never zero). Now, given an arbitrary element $(R + Sy)\omega \in \Omega^1_{A/k}$, with $R, S \in k[x]$, it follows from the above formula that $Sy\omega$ is exact (take $Q = 0$ and P a primitive of S); thus we can write

$$(R + Sy)\omega = R\omega + \text{exact}.$$

Now, chosing appropriate Q (and $P = 0$), we can inductively kill the leading terms of R until we reach

$$(R + Sy)\omega = (r_0 + r_1 x)\omega + \text{exact}.$$

We conclude that

$$H^1_{\mathrm{dR}}(E) = k[dx/y] \oplus k[xdx/y].$$

Remark 4 (Hodge filtration)**.** In general, the de Rham cohomology $H^n_{\mathrm{dR}}(X)$ has a descending filtration by k-subspaces defined by

$$F^p = \mathrm{im}(\mathbb{H}^n(X, \sigma_{\geq p}\Omega^\bullet_{X/k}) \longrightarrow \mathbb{H}^n(X, \Omega^\bullet_{X/k}))$$

where $\sigma_{\geq p}\Omega^\bullet_{X/k} = 0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \Omega^p_{X/k} \longrightarrow \Omega^{p+1}_{X/k} \longrightarrow \cdots$ is the 'stupid truncation' of $\Omega^\bullet_{X/k}$ at the pth degree. For an elliptic curve E, we have

$$F^0 = H^1_{\mathrm{dR}}(E), \qquad F^1 = \Gamma(E, \Omega^1_{E/k}), \qquad F^2 = 0.$$

Let X be a smooth algebraic variety over a subfield k of $\mathbb{C}$. A theorem of Grothendieck [2] says that, after base change to $\mathbb{C}$, the algebraic de Rham cohomology 'computes' the usual singular cohomology of the corresponding complex manifold $X(\mathbb{C})$. Namely, there is a canonical *comparison isomorphism*

$$\mathrm{comp} : H^n_{\mathrm{dR}}(X) \otimes_k \mathbb{C} \xrightarrow{\sim} H^n(X(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = \mathrm{Hom}(H_n(X(\mathbb{C}), \mathbb{Z}), \mathbb{C})$$

which associates the class of an algebraic differential form ω to the integration $\sigma \longmapsto \int_\sigma \omega$. When k is an algebraic number field, the numbers $\int_\sigma \omega$ are called *periods* whenever ω is defined over k and σ is defined over $\mathbb{Q}$.

Example 6. Let E be an elliptic curve given by an affine equation $y^2 = 4x^3 - ux - v$, with $u, v \in \mathbb{C}$ algebraic over $\mathbb{Q}$ satisfying $u^3 - 27v^2 \neq 0$. Given a $\mathbb{Z}$-basis (γ_1, γ_2) of $H_1(E(\mathbb{C}), \mathbb{Z})$, we obtain a *period matrix* representing the comparison isomorphism

$$P = \begin{pmatrix} \int_{\gamma_1} dx/y & \int_{\gamma_1} xdx/y \\ \int_{\gamma_2} dx/y & \int_{\gamma_2} xdx/y \end{pmatrix} = \begin{pmatrix} \omega_1 & \eta_1 \\ \omega_2 & \eta_2 \end{pmatrix}.$$

The integrals of the first kind ω_1, ω_2 are classically called *periods of E*, and generate a lattice $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ such that $\mathbb{C}/\Lambda \cong E(\mathbb{C})$ (see Theorem 1.2). The integrals of the second kind η_1, η_2 are classically called *quasi-periods of E*. Under the modern terminology, they are all periods.

The algebraic de Rham cohomology behaves well in smooth families. If $p : X \longrightarrow S$ is a smooth k-morphism between smooth k-schemes, we define the nth *relative de Rham cohomology* by

$$H^n_{\mathrm{dR}}(X/S) \coloneqq \mathbb{R}^n p_* \Omega^\bullet_{X/S}.$$

This is a quasi-coherent sheaf on S. If p is also proper, such as a family of elliptic curves, then $H^n_{\mathrm{dR}}(X/S)$ is a vector bundle (locally free sheaf of finite rank) whose formation commutes with every base change in S. This means that, given a Cartesian square

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & \square & \downarrow \\ S' & \xrightarrow[f]{} & S \end{array}$$

there's a canonical isomorphism $H^n_{\mathrm{dR}}(X'/S') \xrightarrow{\sim} f^* H^n_{\mathrm{dR}}(X/S)$. In particular, the fiber of the relative de Rham cohomology at a geometric point s of S is simply the de Rham cohomology of the fiber X_s:

$$H^n_{\mathrm{dR}}(X/S)(s) = H^n_{\mathrm{dR}}(X_s).$$

Moreover, the algebraic vector bundle $H^n_{\mathrm{dR}}(X/S)$ commes equipped with the *Gauss-Manin connection*:

$$\nabla : H^n_{\mathrm{dR}}(X/S) \longrightarrow H^n_{\mathrm{dR}}(X/S) \otimes_{\mathcal{O}_S} \Omega^1_{S/k}$$

which is a k-linear map satisfying the Leibniz rule: for local sections g of $\mathcal{O}_S$ and α of $H^n_{\mathrm{dR}}(X/S)$, we have

$$\nabla(g\alpha) = g\nabla(\alpha) + \alpha \otimes dg.$$

Remark 5. Given a vector field (= derivation) D on S, i.e., a global section of $(\Omega^1_{S/k})^\vee \cong \mathcal{D}er_k(\mathcal{O}_S)$, we obtain a k-linear map

$$\nabla_D = (\mathrm{id} \otimes D) \circ \nabla : H^n_{\mathrm{dR}}(X/S) \longrightarrow H^n_{\mathrm{dR}}(X/S)$$

satisfying

$$\nabla_D(g\alpha) = g\nabla_D(\alpha) + D(g)\alpha.$$

Intuitively, if we interpret a section α of $H^n_{\mathrm{dR}}(X/S)$ as a family of cohomology classes $\alpha(s) \in H^n_{\mathrm{dR}}(X_s)$, for $s \in S$, the image $\nabla_D(\alpha)$ is measuring the "infinitesimal variation of α in the direction of D".

To explain where the Gauss-Manin connection comes from, let us assume that $k = \mathbb{C}$ and let us consider the corresponding map $f^{\mathrm{an}} : X(\mathbb{C}) \longrightarrow S(\mathbb{C})$ of complex manifolds. Grothendendieck's comparison theorem also works in families. Let $H^n_{\mathrm{dR}}(X/S)^{\mathrm{an}}$ be the holomorphic vector bundle on the complex manifold $S(\mathbb{C})$ given by 'analytification' of $H^n_{\mathrm{dR}}(X/S)$, and $R^n f^{\mathrm{an}}_* \mathbb{Z}_{X(\mathbb{C})}$ be the local system on $S(\mathbb{C})$ whose fiber at s is the singular cohomology $H^n(X_s(\mathbb{C}), \mathbb{Z})$. Then, the comparison isomorphisms on the fibers glue to a comparison isomorphism

$$\mathrm{comp} : H^n_{\mathrm{dR}}(X/S)^{\mathrm{an}} \xrightarrow{\sim} R^n f^{\mathrm{an}}_* \mathbb{Z}_{X(\mathbb{C})} \otimes \mathcal{O}_{S(\mathbb{C})}$$

identifying the Gauss-Manin connection ∇ with $\mathrm{id} \otimes d$. Since the comparion isomorphism is given by integration of differential forms, the Gauss-Manin connection is simply telling how to "differentiate under the sign of integral". Formally, if α is a section of $H^n_{\mathrm{dR}}(X/S)^{\mathrm{an}}$ and σ a section of $R_n f^{\mathrm{an}}_* \mathbb{Z}_{X(\mathbb{C})} = (R^n f^{\mathrm{an}}_* \mathbb{Z}_{X(\mathbb{C})})^\vee$ (i.e., a locally constant family of topological n-cycles on the fibers $X_s(\mathbb{C})$), then

$$d\left(\int_\sigma \alpha\right) = \int_\sigma \nabla(\alpha)$$

where $\int_\sigma \alpha$ is the function $s \longmapsto \int_{\sigma(s)} \alpha(s)$.

Remark 6. There is an evident analog of the algebraic de Rham cohomology in the category of complex manifolds: the *analytic de Rham cohomology*, where we replace the sheaves of algebraic differential forms by those of holomorphic differential forms. The above comparison isomorphism holds more generally in this context, and it is a theorem that if we start with an algebraic family $f : X \longrightarrow S$, then the analytification $H^n_{\mathrm{dR}}(X/S)^{\mathrm{an}}$ coincides with the analytic de Rham cohomology $\mathbb{R}^n f^{\mathrm{an}}_* \Omega^\bullet_{X(\mathbb{C})/S(\mathbb{C})}$.

We now compute the Gauss-Manin connection on the family $\mathbb{X} \longrightarrow \mathbb{H}$.

Theorem 4.3. *Let $E_2 : \mathbb{H} \longrightarrow \mathbb{C}$ be the holomorphic function defined by*

$$E_2(\tau) = 1 - 24 \sum_{n\geq 1} \sigma_1(n) q^n, \qquad q = e^{2\pi i \tau},$$

and η be the global section of $H^1_{\mathrm{dR}}(\mathbb{X}/\mathbb{H})$ defined by

$$\eta(\tau) = \frac{1}{2\pi i} \wp_\tau(z) dz \in H^1_{\mathrm{dR}}(\mathbb{X}_\tau).$$

If $\nabla : H^1_{\mathrm{dR}}(\mathbb{X}/\mathbb{H}) \longrightarrow H^1_{\mathrm{dR}}(\mathbb{X}/\mathbb{H}) \otimes \Omega^1_{\mathbb{H}}$ denotes the Gauss-Manin connection, then

$$\nabla_D \omega = \eta - \frac{E_2}{12}\omega, \qquad \nabla_D(\nabla_D \omega) = 0$$

where $D = \frac{1}{2\pi i}\frac{d}{d\tau}$.

Note that, under the Weierstrass isomorphism

$$\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z}) \xrightarrow{\sim} E_{(E_4(\tau)/12, -E_6(\tau)/216)}(\mathbb{C}), \qquad z \longmapsto \begin{cases} ((\frac{1}{2\pi i})^2 \wp_\tau(z) : (\frac{1}{2\pi i})^3 \wp'_\tau(z) : 1) & \text{if } z \neq 0 \\ (0:1:0) & \text{if } z = 0 \end{cases},$$

the trivialisation (ω, η) of $H^1_{\mathrm{dR}}(\mathbb{X}/\mathbb{H})$ corresponds to $(dx/y, xdx/y)$.

Proof. Consider the trivialisation (γ_1, γ_2) of the local system $R_1 p_* \mathbb{Z}_{\mathbb{X}}$ such that $(\gamma_{1,\tau}, \gamma_{2,\tau}) = (1, \tau)$ under the identification $H_1(\mathbb{X}_\tau, \mathbb{Z}) \cong \mathbb{Z} + \mathbb{Z}\tau$. To prove that $\nabla_D \omega = \eta - \frac{E_2}{12}\omega$, it suffices to show that

$$\int_{\gamma_i} \nabla_D \omega = \int_{\gamma_i} \eta - \frac{E_2}{12} \int_{\gamma_i} \omega, \qquad i = 1, 2. \tag{2}$$

By definition of the Gauss-Manin connection, we have

$$\int_{\gamma_i} \nabla_D \omega = D(\int_{\gamma_i} \omega) = \begin{cases} 0 & \text{if } i = 1 \\ 1 & \text{if } i = 2 \end{cases}, \tag{3}$$

so that (2) amounts to the equations

$$\int_0^1 \wp_\tau(z) dz = \frac{(2\pi i)^2}{12} E_2(\tau), \qquad \int_0^\tau \wp_\tau(z) dz = 2\pi i + \frac{(2\pi i)^2}{12} E_2(\tau)\tau.$$

These identities are 'classical' (see [3] Lemma A1.3.9 and Remark A1.3.12, and [7] p. 95-96). Finally, $\nabla_D(\nabla_D \omega) = 0$ is an immediate consequence of the fact that $\nabla_D \omega$ is constant (3). □

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