# A crash course in modular forms and cohomology - Lecture 3

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## Contents



## 1 The cohomology $H^1_{\mathrm{dR}}(\mathcal{M}, \mathrm{Sym}^{k-2}\mathcal{H})$

We've seen in the last lecture that to every family of elliptic curves $p : E \longrightarrow S$ we can associate the relative de Rham cohomology $H^1_{\mathrm{dR}}(E/S)$. Moreover, the formation of the relative de Rham cohomology is compatible with every base change in $S$. This allows us to make the following definition.

**Definition 1.1.** The *de Rham bundle* over $\mathcal{M}$ is the quasi-coherent sheaf $\mathcal{H}$ over $\mathcal{M}$ such that $\varphi^*\mathcal{H} = H^1_{\mathrm{dR}}(E/S)$ for a morphism $\varphi : S \longrightarrow \mathcal{M}$ corresponding to a family of elliptic curves $p : E \longrightarrow S$.

Note that the Hodge bundle $\mathcal{F}$ is a subsheaf of $\mathcal{H}$: given $\varphi : S \longrightarrow \mathcal{M}$ as above, $\varphi^*\mathcal{F} = p_*\Omega^1_{E/S}$ is the first step of the Hodge filtration on $\varphi^*\mathcal{H} = H^1_{\mathrm{dR}}(E/S)$.

**Remark 1.** Global sections of tensor powers of the Hodge bundle correspond to modular forms, what can we say about $\mathrm{Sym}^k\mathcal{H}$? Note that $\mathcal{F}^{\otimes k}$ is a subbundle of the symmetric power $\mathrm{Sym}^k\mathcal{H}$. In fact, global sections of $\mathrm{Sym}^k\mathcal{H}$ are related to quasimodular forms (which include modular forms). Consider the Eisentein series

$$E_2(\tau) = 1 - 24\sum_{n\geq 1}\sigma_1(n)q^n, \qquad q = e^{2\pi i\tau}.$$

It's a quasimodular form:

$$E_2(\gamma\tau) = (c\tau + d)^2 E_2(\tau) + \frac{12c}{2\pi i}(c\tau + d), \qquad \gamma \in \mathrm{SL}_2(\mathbb{Z}).$$

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Recall from the second lecture that we can trivialise $\pi^*\mathcal{H}$ by $\omega$ and $\nabla_D\omega$. They satisfy the following transformation rules with respect to the action of $\mathrm{SL}_2(\mathbb{Z})$:

$$\gamma^*\omega = (c\tau+d)^{-1}\omega, \qquad \gamma^*(\nabla_D\omega) = (c\tau+d)\nabla_D\omega - \frac{c}{2\pi i}\omega.$$

One can check that $E_2\omega^2 + 12\omega\nabla_D\omega$ is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$, so that there exists a unique $s \in \Gamma(\mathcal{M}, \mathrm{Sym}^2\mathcal{H})$ such that

$$\pi^*s = E_2\omega^2 + 12\omega\nabla_D\omega.$$

In general, if $s$ is any global section of $\mathrm{Sym}^k\mathcal{H}$, we can write

$$\pi^*s = f_k\omega^k + f_{k-1}\omega^{k-1}\nabla_D\omega + \cdots + f_0(\nabla_D\omega)^k$$

where $f_k$ is a quasimodular form of weight $k$ which completely determines $s$.

We have also seen that the relative de Rham cohomology is equipped with the Gauss-Manin connection. To define it over $\mathcal{M}$, we need the following lemma. Let $\pi_n : M_n \longrightarrow \mathcal{M}$ be the morphism given by the modular curve $M_n$ as defined in the last lecture.

**Lemma 1.2.** *There is a unique quasi-coherent sheaf $\Omega^1_{\mathcal{M}/\mathbb{Q}}$ over $\mathcal{M}$ such that $\pi_n^*\Omega^1_{\mathcal{M}/\mathbb{Q}} = \Omega^1_{M_n/\mathbb{Q}}$ for every $n \geq 3$.*

*Proof.* Since $\mathcal{M} = [\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})\backslash M_n]$, given a morphism $\varphi : S \longrightarrow \mathcal{M}$, there is an open covering $S = \bigcup_i S_i$ and $\varphi_i : S_i \longrightarrow M_n$ such that $\pi_n \circ \varphi_i \cong \varphi$. Since the sheaf $\Omega^1_{M_n/\mathbb{Q}}$ is stable under the action of $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$, we can glue the sheaves $\varphi_i^*\Omega^1_{M_n/\mathbb{Q}}$ over $S_i$ to a sheaf $\varphi^*\Omega^1_{\mathcal{M}/\mathbb{Q}}$ over $S$. □

Since the formation of the Gauss-Manin connection is also compatible with base change, the Gauss-Manin connection $\nabla : H^1_{\mathrm{dR}}(E/M_n) \longrightarrow H^1_{\mathrm{dR}}(E/M_n) \otimes \Omega^1_{M_n/\mathbb{Q}}$ 'descends' to a morphism

$$\nabla : \mathcal{H} \longrightarrow \mathcal{H} \otimes \Omega^1_{\mathcal{M}/\mathbb{Q}}$$

Note that this is not $\mathcal{O}_M$-linear, but only $\mathbb{Q}$-linear.

We can also perform multilinear operations on connections, so we also have a connection on symmetric powers of $\mathcal{H}$. In what follows, we want to compute the following space:

$$H^1_{\mathrm{dR}}(\mathcal{M}, \mathrm{Sym}^{k-2}\mathcal{H}) \coloneqq \mathrm{coker}(\mathrm{Sym}^{k-2}\mathcal{H} \xrightarrow{\nabla} \mathrm{Sym}^{k-2}\mathcal{H} \otimes \Omega^1_{\mathcal{M}/\mathbb{Q}})$$

**Remark 2.** The $\mathbb{Q}$-vector space $H^1_{\mathrm{dR}}(\mathcal{M}, \mathrm{Sym}^{k-2}\mathcal{H})$ can also be recovered as the $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$-invariant subspace of

$$H^1_{\mathrm{dR}}(M_n, \pi_n^*\mathrm{Sym}^{k-2}\mathcal{H}) \coloneqq \mathrm{coker}(\mathrm{Sym}^{k-2}H^1_{\mathrm{dR}}(E/M_n) \xrightarrow{\nabla} \mathrm{Sym}^{k-2}H^1_{\mathrm{dR}}(M_n/\mathbb{Q}) \otimes \Omega^1_{M_n/\mathbb{Q}})$$

In general, given a smooth scheme $X$ over a field $k$ and a vector bundle with integrable connection $(\mathcal{E}, \nabla)$ on $X$, the $n$th algebraic de Rham cohomology with coefficients in $\mathcal{E}$ is the $k$-vector space $H^n_{\mathrm{dR}}(X, \mathcal{E}) = \mathbb{H}^n(X, \mathcal{E} \otimes \Omega^\bullet_{X/k})$, where $\mathcal{E} \otimes \Omega^\bullet_{X/k}$ is a complex induced by $\nabla$ (see [5]). If $X$ is affine, than $H^n_{\mathrm{dR}}(X, \mathcal{E})$ boils down to the cohomology of the complex of global sections $\Gamma(X, \mathcal{E} \otimes \Omega^\bullet_{X/k})$. This explains our ad-hoc definition above.

For the next theorem, let $M_k^! = \Gamma(\mathcal{M}, \mathcal{F}^{\otimes k})$ denote the $\mathbb{Q}$-space of weakly holomorphic modular forms of weight $k$ over $\mathbb{Q}$. Recall that these are holomorphic functions $f : \mathbb{H} \longrightarrow \mathbb{C}$, modular of weight $k$, and whose Fourier series is of the form

$$f(\tau) = \sum_{n \gg -\infty} a_n q^n, \qquad q = e^{2\pi i\tau}$$

with $a_n \in \mathbb{Q}$.

**Theorem 1.3.** *For every even $k \geq 2$, we have*

$$H^1_{\mathrm{dR}}(\mathcal{M}, \mathrm{Sym}^{k-2}\mathcal{H}) \cong M^!_k / D^{k-1} M^!_{2-k}$$

*where* $D = \frac{1}{2\pi i}\frac{d}{d\tau} = q\frac{d}{dq}$.

In general, the derivative of a modular form is not modular: if $f(\gamma\tau) = (c\tau + d)^k f(\tau)$, then

$$Df(\gamma\tau) = (c\tau + d)^{k+2} Df(\tau) + 2\pi i k c (c\tau + d)^{k+1} f(\tau).$$

It is true however that, if $f \in M^!_{2-k}$, then $D^{k-1}f = D(D(\cdots D(f)\cdots))$ is modular of weight $k$. This is a consequence of the so-called 'Bol's identity', but it also follows from the proof of the above theorem.

Before proving Theorem 1.3, we consider some examples. For this, we will also use the following notation:

- $M_k$ is the space of modular forms of weight $k$ over $\mathbb{Q}$,
- $S^!_k \leq M^!_k$ is the subspace of weakly holomorphic cuspforms of weight $k$ over $\mathbb{Q}$, defined by the condition $a_0(f) = 0$,
- $S_k = M_k \cap S^!_k$ is the space of cuspforms of weight $k$ over $\mathbb{Q}$.

We shall also consider the normalised Eisenstein series ($k \geq 4$ even)

$$E_k(\tau) = (2\zeta(k))^{-1} G_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n \geq 1} \sigma_{k-1}(n) q^n \in M_k$$

and Ramanujan's delta:

$$\Delta(\tau) = \frac{1}{1728}(E_4(\tau)^3 - E_6(\tau)^2) = q \prod_{n \geq 1} (1 - q^n)^{24} \in S_{12}$$

This product formula for the $q$-expansion of $\Delta$ is a theorem (see [8] Chapter 7, Theorem 6, or [9] 2.4). It implies in particular that $\Delta$ has no zeroes on $\mathbb{H}$, so we have, for instance,

$$\Delta^{-1} \in M^!_{-12}.$$

The basic tool for studying spaces of modular forms is the following 'valence formula'.

**Theorem 1.4.** *For every $f \in M^!_k$, we have*

$$\mathrm{ord}_\infty(f) + \sum_{p \in \mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}} \frac{\mathrm{ord}_p(f)}{e_p} = \frac{k}{12}$$

*where $e_p = 2$ (resp. $e_p = 3$) if $p = \mathrm{SL}_2(\mathbb{Z}) \cdot i$ (resp. $p = \mathrm{SL}_2(\mathbb{Z}) \cdot e^{\frac{2\pi i}{3}}$), and $e_p = 1$ otherwise.*

For a proof, see [8] Chapter 7, Theorem 3, or [9] Proposition 2. Under an appropriate framework, this can be seen as a consequence of the Riemann-Roch formula for tensor powers of the Hodge bundle over the compactification $\overline{\mathcal{M}}$.

**Example 1** $(k=2)$**.** We have

$$H^1_{\mathrm{dR}}(\mathcal{M}) = H^1_{\mathrm{dR}}(\mathcal{M}, \mathrm{Sym}^0\mathcal{H}) = M_2^!/DM_0^! = 0.$$

Note that

$$j(\tau) = \frac{E_4(\tau)^3}{\Delta(\tau)} = \frac{1}{q} + 744 + 196884q + \cdots \in M_0^!.$$

In fact, it follows from the valence formula that

$$M_0^! = \mathbb{Q}[j].$$

To see this, note that $j^n \in M_0^!$ and $\mathrm{ord}_\infty(j^n) = -n$ for every $n \geq 0$. If $f \in M_0^!$ has a pole of order $n$ at infinity, by solving a linear system, we can find a polynomial $P(j) \in \mathbb{Q}[j]$ of degree $n$ such that $f - P(j)$ is holomorphic at infinity. But it follows from the valence formula that $M_0 = 0$, so that $f = P(j)$.

Note that $Dj = -q^{-1} + 196884q + \cdots$ is in $M_2^!$. Again, by the valence formula, we have $M_2 = 0$, and we conclude similarly that

$$M_2^! = \mathbb{Q}[Dj] = DM_0^!$$

This proves that $H^1_{\mathrm{dR}}(\mathcal{M}, \mathrm{Sym}^0\mathcal{H}) = 0$.

**Example 2** $(k=4)$**.** We have

$$H^1_{\mathrm{dR}}(\mathcal{M}, \mathrm{Sym}^2\mathcal{H}) = M_4^!/D^3M_{-2}^! = \mathbb{Q}\cdot[E_4]$$

By the valence formula, we have $S_4 = 0$, so that $M_4 = \mathbb{Q}\cdot[E_4]$. For every $n \geq 1$, we can construct an element of $M_{-2}^!$ having a pole of order $n$ at infinity, namely, $E_{12n-2}/\Delta^n$. Thus, given any $f \in M_4^!$ with $\mathrm{ord}_\infty(f) = -n$, we can find a $\lambda_i \in \mathbb{Q}$ such that

$$f - \sum_{i=1}^{n} \lambda_i D^3(E_{12i-2}/\Delta^i) \in M_4 = \mathbb{Q}E_4$$

On the other hand, $E_4$ is not in the image of $D^3$ since it $a_0(E_4) \neq 0$. This proves that $M_4^! = \mathbb{Q}E_4 \oplus D^3M_{-2}^!$.

Similarly, we prove that $H^1_{\mathrm{dR}}(\mathcal{M}, \mathrm{Sym}^{k-2}\mathcal{H}) = \mathbb{Q}\cdot[E_k]$ for $k = 6, 8, 10, 14$, because in these cases $S_k = 0$ (again, by the valence formula).

**Example 3** $(k=12)$**.** We have

$$H^1_{\mathrm{dR}}(\mathcal{M}, \mathrm{Sym}^{10}\mathcal{H}) = M_{12}^!/D^{11}M_{-10}^! = \mathbb{Q}\cdot[E_{12}] \oplus S_{12}^!/D^{11}M_{-10}^!$$

and

$$S_{12}^!/D^{11}M_{-10}^! = \mathbb{Q}\cdot[\Delta] \oplus \mathbb{Q}\cdot[\Delta']$$

where

$$\Delta' = \frac{1}{q} + 47709536q^2 + 39862705122q^3 + 7552626810624q^4 + \cdots$$

is the unique element of $S_{12}^!$ having a simple pole at infinity and such that $a_1 = 0$. To construct $\Delta'$, consider $jE_{12}$, then add to it a suitable combination of $E_{12}$ and $\Delta$. To see that the image of $\Delta$ and $\Delta'$ in the quotient $S_{12}^!/D^{11}M_{-10}^!$ generate it as a $\mathbb{Q}$-vector space, we note that for every $n \geq 2$, we can construct an element of $D^{11}M_{-10}^!$ with a pole of order $n$ at infinity, namely, $D^{11}(E_{12n-10}/\Delta^n)$, and we argue as in the last example. Finally, $(\mathbb{Q}\Delta + \mathbb{Q}\Delta') \cap D^{11}M_{-10}^!$ because the valence formula guarantees that there's no element of $M_{-10}^!$ having at most a simple pole at infinity.

In general, we have

$$M_k^!/D^{k-1}M_{2-k}^! = \mathbb{Q}\cdot[E_k]\oplus S_k^!/D^{k-1}M_{2-k}^!$$

The subspace $S_k^!/D^{k-1}M_{2-k}^!$ of $M_k^!/D^{k-1}M_{2-k}^!$ also has a geometric interpretation in terms of cuspidal (or parabolic) cohomology:

$$H^1_{\mathrm{dR,cusp}}(\mathcal{M},\mathrm{Sym}^{k-2}\mathcal{H})\cong S_k^!/D^{k-1}M_{2-k}^!$$

This cohomology space shares some similarities with the first cohomology of a smooth projective curve.

**Theorem 1.5.** *Let $k\geq 2$ be an even integer.*

1. *We have*

$$\dim_{\mathbb{Q}} S_k^!/D^{k-1}M_{2-k}^! = 2\dim_{\mathbb{Q}} S_k < +\infty$$

2. *The pairing*

$$S_k^!\times S_k^!\longrightarrow\mathbb{Q},\qquad (f,g)\longmapsto\sum_{n\in\mathbb{Z}}\frac{a_n(f)a_{-n}(g)}{n^{k-1}}$$

*induces a symplectic $\mathbb{Q}$-bilinear pairing on $S_k^!/D^{k-1}M_{2-k}^!$ for which $S_k\hookrightarrow S_k^!/D^{k-1}M_{2-k}^!$ is an isotropic subspace.*

*Proof.* See [4] for a direct elementary proof. Using that $S_k^!/D^{k-1}M_{2-k}^!$ is a cohomology group, this statement can also be deduced from general geometric considerations. □

The analogy with the cohomology of a smooth projective curve $X$ is that $S_k$ corresponds to the subspace of holomorphic forms $H^0(X,\Omega^1)\subset H^1_{\mathrm{dR}}(X)$ (the Hodge filtration), and the above pairing corresponds to the cup product. For instance, for an elliptic curve $E$, the decomposition of

$$H^1_{\mathrm{dR}}(E)=\mathbb{Q}\cdot[dx/y]\oplus\mathbb{Q}\cdot[xdx/y]$$

in terms of forms of the 'first kind' $dx/y$ and of the 'second kind' $xdx/y$ corresponds to the decomposition

$$H^1_{\mathrm{dR,cusp}}(\mathcal{M},\mathrm{Sym}^{10}\mathcal{H})=\mathbb{Q}\cdot[\Delta]\oplus\mathbb{Q}\cdot[\Delta']$$

of Example 3.

**Remark 3.** Formally, one can show that $H^1_{\mathrm{dR,cusp}}(\mathcal{M},\mathrm{Sym}^{k-2}\mathcal{H})\cong S_k^!/D^{k-1}M_{2-k}^!$ underlies a polarisable pure Hodge of type $\{(k-1,0),(0,k-1)\}$ such that $F^{k-1}\cong S_k$.

## 2 Proof of the main theorem

We want to prove

$$\mathrm{coker}(\mathrm{Sym}^{k-2}\mathcal{H}\xrightarrow{\nabla}\mathrm{Sym}^{k-2}\mathcal{H}\otimes\Omega^1_{\mathcal{M}/\mathbb{Q}})\cong\mathrm{coker}(M_{2-k}^!\xrightarrow{D^{k-1}}M_k^!)\tag{1}$$

Our proof will be based on the Kodaira-Spencer isomorphism. To explain it, we need the *de Rham pairing*

$$\langle\ ,\ \rangle_{\mathrm{dR}}:\mathcal{H}\otimes\mathcal{H}\longrightarrow\mathcal{O}_{\mathcal{M}}$$

which is an alternating and perfect $\mathcal{O}_{\mathcal{M}}$-linear pairing on $\mathcal{H}$, defined in terms of the de Rham cup product. It is characterised by the following property. If $\varphi:S\longrightarrow\mathcal{M}$ is a morphism correponding to a family of elliptic curves $p:E\longrightarrow S$ which admits a Weierstrass equation, then

$$\langle dx/y, xdx/y\rangle_{\mathrm{dR}}=1.$$

**Theorem 2.1.** *The map*

$$\kappa : \mathcal{F}^{\otimes 2} \xrightarrow{\sim} \Omega^1_{\mathcal{M}/\mathbb{Q}}, \qquad s_1 \otimes s_2 \longmapsto \langle s_1, \nabla s_2 \rangle_{\mathrm{dR}}$$

*is an isomorphism of quasi-coherent sheaves on $\mathcal{O}_{\mathcal{M}}$.*

*Proof.* To see that $\kappa$ is indeed $\mathcal{O}_{\mathcal{M}}$-linear, we compute

$$\langle s_1, \nabla(g s_2) \rangle_{\mathrm{dR}} = \langle s_1, s_2 \otimes dg + g \nabla s_2 \rangle_{\mathrm{dR}} = \langle s_1, s_2 \rangle_{\mathrm{dR}} \otimes dg + g \langle s_1, \nabla s_2 \rangle_{\mathrm{dR}}$$

Note that $\langle s_1, s_2 \rangle_{\mathrm{dR}} = 0$ because $\langle\ ,\ \rangle_{\mathrm{dR}}$ is alternating and both $s_1$ and $s_2$ are sections of the same line bundle $\mathcal{F} \subset \mathcal{H}$. To check that $\kappa$ is an isomorphism, it suffices to do it after pulling back to $\mathbb{H}$ via $\pi : \mathbb{H} \longrightarrow \mathcal{M}^{\mathrm{an}}$. Recall from last lecture that $\nabla_D \omega = \eta - \frac{E_2}{12}\omega$, and that $\omega$ and $\eta$ correspond to $dx/y$ and $xdx/y$ for some Weierstrass equation for $\mathbb{X} \longrightarrow \mathbb{H}$. Thus $\langle \omega, \nabla_D \omega \rangle_{\mathrm{dR}} = 1$, and we have

$$\langle \omega, \nabla \omega \rangle_{\mathrm{dR}} = 2\pi i \, d\tau$$

As $\omega$ trivialises $\pi^* \mathcal{F}$ and $2\pi i \, d\tau$ trivialises $\Omega^1_{\mathcal{H}}$, $\kappa$ is an isomorphism. ☐

It follows from the Kodaira-Spence isomorphism that

$$\begin{aligned} M_k^! &\xrightarrow{\sim} \Gamma(\mathcal{M}, \mathcal{F}^{\otimes k-2} \otimes \Omega^1_{\mathcal{M}/\mathbb{Q}}) \subset \Gamma(\mathcal{M}, \mathrm{Sym}^{k-2} \mathcal{H} \otimes \Omega^1_{\mathcal{M}/\mathbb{Q}}) \\ f &\longmapsto f \omega^{\otimes k-2} \otimes 2\pi i \, d\tau \end{aligned}$$

Thus, (1) is equivalent to the following assertions:

1. $\Gamma(\mathcal{M}, \mathcal{F}^{\otimes k-2} \otimes \Omega^1) + \mathrm{im}\, \nabla = \Gamma(\mathcal{M}, \mathrm{Sym}^{k-2} \mathcal{H} \otimes \Omega^1)$
2. $\Gamma(\mathcal{M}, \mathcal{F}^{\otimes k-2} \otimes \Omega^1) \cap (\mathrm{im}\, \nabla) \cong D^{k-1} M^!_{2-k}$

To prove 1, we use the following lemma.

**Lemma 2.2.** *The Gauss-Manin connection induces isomorphisms*

$$\nabla : F^p / F^{p+1} \xrightarrow{\sim} F^{p-1}/F^p \otimes \Omega^1$$

*In particular,*

$$\nabla : F^1 \xrightarrow{\sim} F^0 / F^{k-2} \otimes \Omega^1$$

*Proof.* It suffices to prove the corresponding statement after pulling back to $\mathbb{H}$. Since $\omega^p (\nabla_D \omega)^{k-2-p} + \pi^* F^{p+1}$ trivialises $\pi^*(F^p/F^{p+1})$ and that

$$\nabla(\omega^p (\nabla_D \omega)^{k-2-p}) = \nabla_D(\omega^p (\nabla_D \omega)^{k-2-p}) \otimes 2\pi i \, d\tau = p \omega^{p-1} (\nabla_D \omega)^{k-1-p}) \otimes 2\pi i \, d\tau,$$

we conclude that $\nabla$ sends a trivialisation of $\pi^*(F^p/F^{p+1})$ to a trivialisation of $\pi^*(F^{p-1}/F^p) \otimes \Omega^1_{\mathbb{H}}$, so that it is an isomorphism. The last statement follows by considering the splitting $\pi^* F^1 = \pi^*(F^1/F^2) \oplus \cdots \oplus \pi^*(F^{k-3}/F^{k-2}) \oplus \pi^* F^{k-2}$ given by $(\omega, \nabla_D \omega)$. ☐

Thus, given $\alpha \in \Gamma(\mathcal{M}, F^0 \otimes \Omega^1)$, there's $\beta \in \Gamma(\mathcal{M}, F^{k-2} \otimes \Omega^1)$ and $s \in \Gamma(\mathcal{M}, F^1)$ such that

$$\alpha = \beta + \nabla s$$

This proves the statement 1 above.

To prove 2, let $s \in \Gamma(\mathcal{M}, \operatorname{Sym}^{k-2}\mathcal{H})$ be such that $\nabla s \in \Gamma(\mathcal{M}, F^{k-2} \otimes \Omega^1)$. By pulling back to $\mathbb{H}$, we get an equation of the form

$$\nabla(s_{k-2}\omega^{k-2} + s_{k-3}\omega^{k-3}\nabla_D\omega + \cdots + s_0(\nabla_D\omega)^{k-2}) = f\omega^{k-2} \otimes 2\pi i\, d\tau \tag{2}$$

where $s_i : \mathbb{H} \longrightarrow \mathbb{C}$ are holomorphic functions and $f : \mathbb{H} \longrightarrow \mathbb{C}$ is a weakly holomorphic modular form of weight $k$. Equation (2) is equivalent to

$$\nabla_D(s_{k-2}\omega^{k-2} + s_{k-3}\omega^{k-3}\nabla_D\omega + \cdots + s_0(\nabla_D\omega)^{k-2}) = f\omega^{k-2}.$$

By applying the Leibniz rule to the left-hand side, we get

$$D(s_{k-2})\omega^{k-2} + (D(s_{k-3}) + (k-2)s_{k-2})\omega^{k-3}\nabla_D\omega + \cdots + (D(s_1) + s_0)(\nabla_D\omega)^{k-2} = f\omega^{k-2}$$

so that $D(s_{k-2}) = f$ and

$$D(s_{j-1}) + js_j = 0, \qquad j = 1, \ldots, k-2$$

By induction, we get

$$f = \frac{1}{(k-2)!}D^{k-1}(s_0).$$

To finish, we remark that $s_0 \in M^!_{2-k}$. This follows immediately from the fact that $\pi^*s = \sum_i s_i\omega^i(\nabla_D\omega)^{k-2-i}$ is $\mathrm{SL}_2(\mathbb{Z})$-invariant, and from the explicit description of the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\omega$ and $\nabla_D\omega$ given in Remark 1.

**Remark 4.** Alternatively, to see that $s_0 \in M^!_{2-k} \cong \Gamma(\mathcal{M}, (\mathcal{F}^{\otimes k-2})^\vee)$, we note that the pairing $\langle\ ,\ \rangle_{\mathrm{dR}}$ induces an isomorphism

$$F^0/F^{k-3} \xrightarrow{\sim} (F^{k-2})^\vee, \qquad t \longmapsto \langle\ , t\rangle_{\mathrm{dR}}$$

and that $s$ as above is sent to $s_0\langle\ \ , (\nabla_D\omega)^{k-2}\rangle_{\mathrm{dR}} = s_0(\omega^{k-2})^\vee$.

# 3 Periods and the Eichler-Shimura isomorphism

We briefly discuss the relation between the above constructions and the more classical Eichler cohomology groups. For this, consider the $\mathbb{Q}$-vector space of homogeneous polynomials of degree $n$ with $\mathbb{Q}$-coefficients:

$$V_n = \operatorname{Sym}^n(\mathbb{Q}X + \mathbb{Q}Y)$$

with the right $\mathrm{SL}_2(\mathbb{Z})$-action:

$$(X, Y)|_\gamma = (aX + bY, cX + dY), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

We can then consider the group cohomology

$$H^1(\mathrm{SL}_2(\mathbb{Z}), V_n) = \frac{\{c : \mathrm{SL}_2(\mathbb{Z}) \to V_n\ ;\ c(\gamma_1\gamma_2) = c(\gamma_1)|_{\gamma_2} + c(\gamma_2)\}}{\{c : \mathrm{SL}_2(\mathbb{Z}) \to V_n\ ;\ c(\gamma) = v|_\gamma - v, \text{ for some } v \in V_n\}}$$

Given $\tau_0 \in \mathbb{H}$ and $f \in M_k$, we can define a 1-cocyle $c_f : \mathrm{SL}_2(\mathbb{Z}) \longrightarrow V_{k-2}$ by the formula

$$c_f(\gamma) = (2\pi i)^{k-1}\int_{\gamma^{-1}\tau_0}^{\tau_0} f(\tau)(X - \tau Y)^{k-2}d\tau$$

Different choices of $\tau_0$ yield cohomologous cocycles.

**Example 4.** If $f \in S_k$, we can rather can take $\tau_0 = \infty$. The *period polynomial* of $f$ is defined as

$$c_f(S) = (2\pi i)^{k-1} \int_0^\infty f(\tau)(X - \tau Y)^{k-2} d\tau \in \mathbb{C}[X,Y], \qquad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

It's coefficients are the *periods* of $f$ (up to some normalisation):

$$\int_0^\infty f(it) t^m dt, \qquad 0 \le m \le k-2$$

and we have seen in the first lecture (at least when $f = \Delta$) that they compute the special values of $L(f,s)$ at the critical strip.

The theorem of Eichler-Shimura asserts that

$$(M_k \otimes \mathbb{C}) \oplus \overline{(S_k \otimes \mathbb{C})} \xrightarrow{\sim} H^1(\mathrm{SL}_2(\mathbb{Z}), V_{k-2}) \otimes \mathbb{C}, \qquad (f, \overline{g}) \longmapsto c_f + c_{\overline{g}}$$

is an isomorphism of $\mathbb{C}$-vector spaces. In particular,

$$\dim H^1(\mathrm{SL}_2(\mathbb{Z}), V_{k-2}) = \dim H^1_{\mathrm{dR}}(\mathcal{M}, \mathrm{Sym}^{k-2}\mathcal{H}) = 1 + 2\dim S_k$$

and it's natural to ask what is the relation between these two cohomology groups. Geometrically, a vector space with an $\mathrm{SL}_2(\mathbb{Z})$-action gives rise to a local system over $\mathcal{M}^{\mathrm{an}} \cong [\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}]$; the cohomology of $\mathcal{M}^{\mathrm{an}}$ with coefficients in this local system coincides with the group cohomology $H^1(\mathrm{SL}_2(\mathbb{Z}), V_{k-2})$. In other words, $H^1(\mathrm{SL}_2(\mathbb{Z}), V_{k-2})$ is the 'Betti counterpart' of $H^1_{\mathrm{dR}}(\mathcal{M}, \mathrm{Sym}^{k-2}\mathcal{H}) \cong M_k^!/D^{k-1}M_{2-k}^!$. Indeed, there is a comparison isomorphism

$$\mathrm{comp} : (M_k^!/D^{k-1}M_{2-k}^!) \otimes \mathbb{C} \xrightarrow{\sim} H^1(\mathrm{SL}_2(\mathbb{Z}), V_{k-2}) \otimes \mathbb{C}, \qquad f \longmapsto [c_f]$$

Under the this interpretation, the Eichler-Shimura isomorphism is simply describing the Hodge decomposition on $H^1(\mathrm{SL}_2(\mathbb{Z}), V_{k-2}) \otimes \mathbb{C}$.

There's also a cuspidal version

$$\mathrm{comp} : S_k^!/D^{k-1}M_{2-k}^! \otimes \mathbb{C} \xrightarrow{\sim} H^1_{\mathrm{par}}(\mathrm{SL}_2(\mathbb{Z}), V_{k-2}) \otimes \mathbb{C}$$

where $H^1_{\mathrm{par}}(\mathrm{SL}_2(\mathbb{Z}), V_{k-2})$ is given by cocyles that vanish on $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. This 'parabolic cohomology group' is what is most often called the Eichler cohomology.

**Example 5.** Consider the cuspidal comparison isomorphism for $k = 12$. It gives rise to a period matrix of the form

$$\begin{pmatrix} \omega_\Delta^+ & i\eta_\Delta^+ \\ \omega_\Delta^- & i\eta_\Delta^- \end{pmatrix} = \begin{pmatrix} -68916772.809... & i127202100647.177... \\ -5585015.379... & i10276732343.649... \end{pmatrix}$$

The numbers $\omega_\Delta^+$ and $\omega_\Delta^-$ are 'the' periods of $\Delta$. The integrals $\int_0^\infty \Delta(it)t^m dt$ are multiples of $\omega_\Delta^\pm$; in fact, there are $P^+, P^- \in \mathbb{Q}[X,Y]$ such that

$$(2\pi i)^{11} \int_0^\infty \Delta(\tau)(X - \tau Y)^{10} d\tau = \omega_\Delta^+ P^+ + \omega_\Delta^- P^-$$

The numbers $\eta_\Delta^\pm$ can be called the quasi-periods of $\Delta$, in analogy with the elliptic curves terminology. They are not well studied in the literature.

## 4 Final remarks

- The proof we gave for the isomorphism $H^1_{\mathrm{dR}}(\mathcal{M}, \mathrm{Sym}^k \mathcal{H}) \cong M^!_k/D^{k-1}M^!_{2-k}$ is close to the arguments found in [2]. See also the recent papers [6] and [1]. For a more direct study of $M^!_k/D^{k-1}M^!_{2-k}$, see [4].
- Quasi-periods of modular forms seem to have been first introduced in [1].
- $S^!_{12}/D^{11}M^!_{-10}$ and $H^1_{\mathrm{par}}(\mathrm{SL}_2(\mathbb{Z}), V_{10})$ are the de Rham and Betti realisations of a motive $M_\Delta$. In general, Hecke theory splits $S^!_k/D^{k-1}M^!_{2-k}$ and $H^1_{\mathrm{par}}(\mathrm{SL}_2(\mathbb{Z}), V_k)$ into realisations of motives of Hecke cuspforms. For motives of modular forms, see [7].

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