A useful class of group actions, which generalizes actions of compact groups on Hausdorff spaces, are the so-called proper actions. They are particularly useful for locally compact (and for discrete) groups.

First we recall the point set topology of proper maps (see Bourbaki [1961a], § 10). Let $f\colon X \to Y$ be a continuous map. Then f is called **proper** if one of the following equivalent properties holds:

(3.13) For each topological space Z, the map $f \times \mathrm{id}\colon X \times Z \to Y \times Z$ is closed.

(3.14) f is closed and, for each $y \in Y$, the pre-image $f^{-1}(y)$ is compact.

If X and Y are Hausdorff spaces and Y is locally compact, then f is proper if and only if for each compact subset $K \subset Y$, the pre-image $f^{-1}(K)$ is compact. In this case, X is also locally compact.

(3.15) Let $f\colon X \to Y$ be a continuous injective map. Then f is proper if and only if f is closed if and only if f is a homeomorphism onto a closed subspace.

(3.16) Let $f\colon X \to Y$ and $g\colon Y \to Z$ be continuous maps.
(i) If f and g are proper, then gf is proper.
(ii) If gf is proper and f surjective, then g is proper.
(iii) If gf is proper and g injective, then f is proper.

(3.17) Let $f\colon X \to Y$ be continuous and $B \subset Y$. If f is proper, then the induced map $f^{-1}(B) \to B$ is proper.

An **action** $\varrho\colon G \times X \to X$, $(g, x) \mapsto gx$ of the topological group G on the space X is called **proper** if the associated map

$$\theta = \theta_\varrho\colon\ G \times X \to X \times X,\ (g, x) \mapsto (x, gx)$$

is proper in the sense defined above. We are going to collect some properties of proper actions, thereby generalizing some of the previous results.

(3.18) Proposition. *Given a proper action of G on X. Then:*
(i) *X/G is a Hausdorff space.*
(ii) *If G is Hausdorff, then X is Hausdorff.*

Proof.
(i) Let C be the image of θ. Since θ is proper, θ is also closed. Hence $C \subset X \times X$ is closed. The assertion now follows from (3.1, iv) and (3.2).
(ii) $X \to G \times X$, $x \mapsto (e, x)$ is a homeomorphism onto a closed subset and hence is proper by (3.15). The composition with θ is proper (3.16). This composition is the diagonal $X \to X \times X$, which therefore has a closed image. □

The next proposition shows that proper actions have special properties.

(3.19) Proposition. *Let G act properly on X. For each $x \in X$ the following holds:*
(i) *$\omega\colon G \to X$, $g \mapsto gx$ is proper.*
(ii) *The isotropy group G_x is compact.*
(iii) *The map $\omega'\colon G/G_x \to Gx$ induced by ω is a homeomorphism.*
(iv) *The orbit Gx is closed in X.*

Proof. We have $\theta^{-1}(\{x\} \times X) = G \times \{x\}$. Therefore, by (3.17), the map $G \times \{x\} \to \{x\} \times X$, $(g, x) \mapsto (x, gx)$ is proper. This shows (i). The pre-image $\omega^{-1}(x) = G_x$ is compact by (3.14). Since ω is proper, its image Gx is closed in X. The map ω' is proper by (3.16) and therefore a homeomorphism (3.15). □

(3.20) Proposition. *Let G act freely on X. The following are equivalent:*
(i) *G acts properly.*
(ii) *$C = \text{image}\, \theta \subset X \times X$ is closed and $\varphi\colon C \to G$, $(x, gx) \mapsto g$ is continuous.*

Proof. Since G acts freely, the map θ is injective. By (3.15), θ is proper if and only if C is closed and $\theta'\colon G \times X \to C$, $(g, x) \mapsto (x, gx)$ is a homeomorphism. The map $C \to G \times X$, $(x, y) \mapsto (\varphi(x, y), x)$ is inverse to θ'. It is continuous if and only if φ is continuous. Thus θ' is a homeomorphism if and only if φ is continuous. □

The next result is an important characterization of proper actions for locally compact groups.

(3.21) Proposition. *Let the locally compact Hausdorff group G act on the Hausdorff space X. Then G acts properly if and only if the following holds: For each pair x, y of points in X, there exist neighbourhoods V_x of x and V_y of y in X such that $\{g \in G \mid gV_x \cap V_y \neq \emptyset\}$ is relatively compact in G.*

Proof. Suppose the latter condition is satisfied. We show that $\theta\colon G \times X \to X \times X$ is closed. Let $A \subset G \times X$ be closed. Let $((x_j, y_j) \mid j \in J)$ be a net of points in $\theta(A)$ which converges to $(x, y) \in X \times X$. We have to show that $(x, y) \in \theta(A)$.

Write $y_j = g_j x_j$ with $(g_j, x_j) \in A$. Choose V_x and V_y such that $\{g \mid gV_x \cap V_y \neq \emptyset\}$ is contained in a compact set K. We may assume that $x_j \in V_x$, $y_j \in V_y$ for all j. Then $g_j \in K$ and, by compactness, there exists a subnet (g_α) of (g_j) which converges to $g \in K$. Since A is closed, we have $(g, x) \in A$ and since θ is continuous, $\theta(g, x) = (x, y)$. It is also easy to show that θ has compact pre-images of points. Thus θ is proper.

Conversely, assume that θ is proper. Then $G \times X \to G \times X \times X$, $(g, x) \mapsto (g, x, gx)$ is a homeomorphism onto its image D and it transforms the proper map θ to the proper map $p\colon D \to X \times X$, $(g, x, gx) \mapsto (x, gx)$. Let F

$= G \cup \{\infty\}$ be the one-point-compactification of G. We show that D is closed in $F \times X \times X$. The set $E = \{(g, g) | g \in G\} \subset F \times G$ is closed, being the graph of the inclusion $G \to F$. Therefore, $(E \times X \times X) \cap (F \times D) =: U$ is closed in $F \times D$. Since p is proper, u: $F \times D \to F \times X \times X$, $(h, g, x, y) \mapsto (h, x, y)$ is closed. Since $u(D) = D$, we conclude that D is closed in $F \times X \times X$, as claimed. We have $(\{\infty\} \times X \times X) \cap D = \emptyset$. Therefore, there exist neighbourhoods V of $\{\infty\}$ in F and W of (x, y) in $X \times X$ such that $(V \times W) \cap D = \emptyset$. By definition of F, we can take V to be of the form $(G \backslash K) \cup \{\infty\}$, $K \subset G$ compact. If $V_x \times V_y \subset W$, then $((G \backslash K) \times (V_x \times V_y)) \cap D = \emptyset$ which is equivalent to $(g \notin K \Rightarrow gV_x \cap V_y = \emptyset)$. □

(3.22) Corollary. *A discrete group G acts properly on the Hausdorff space X if and only if for each pair of points (x, y) in X there exist neighbourhoods V_x of x and V_y of y such that $\{g | gV_x \cap V_y \neq \emptyset\}$ is finite.* □

In the literature, proper actions of discrete groups are often called **properly discontinuous**.

A proper action of a discrete group has locally the orbit space of a finite group action. The next proposition makes this precise.

(3.23) Proposition. *Let the discrete group G act properly on the Hausdorff space X. Then the isotropy group G_x of $x \in X$ is finite. Moreover:*

(i) *There exists an open neighbourhood U of x which is a G_x-subspace and satisfies $U \cap gU = \emptyset$ for $g \notin G_x$.*

(ii) *U can be chosen in such a way that the canonical map $U/G_x \to X/G$ is a homeomorphism onto an open set.*

Proof. G_x is finite by (3.19). By (3.21), there exists an open neighbourhood U_0 of x such that $K = \{g \in G | gU_0 \cap U_0 \neq \emptyset\}$ is finite. We have $G_x \subset K$. Let $g_1, \ldots, g_n$ be the elements of $K \backslash G_x$. The points $x_i = g_i x$ are different from x. Since X is a Hausdorff space, there exist open neighbourhoods V_i of x and V_i' of $g_i x$ such that $V_i \cap V_i' = \emptyset$. Let $U_i = V_i \cap g_i^{-1} V_i'$. This is an open neighbourhood of x satisfying $U_i \cap g_i U_i \subset V_i \cap V_i' = \emptyset$. Let $U' = U_0 \cap U_1 \cap \ldots \cap U_n$. This open neighbourhood of x satisfies $U' \cap gU' = \emptyset$ for $g \notin G_x$. The neighbourhood $U = \bigcap_{g \in G_x} gU'$ is a G_x-subspace and satisfies $U \cap gU = \emptyset$ for $g \notin G_x$.

The canonical map $U/G_x \to X/G$ is injective by construction. Moreover, it is continuous and open, thus a homeomorphism onto its image. □

(3.24) Corollary. *Let the discrete group G act freely and properly on the Hausdorff space X. Then the orbit map $X \to X/G$ is a covering, i.e. a locally trivial map with typical fibre G.* □

(3.25) Exercises. In these exercises, compact groups are assumed to be Hausdorff.